Workshop on pseudorandom unitaries

Fermi Ma

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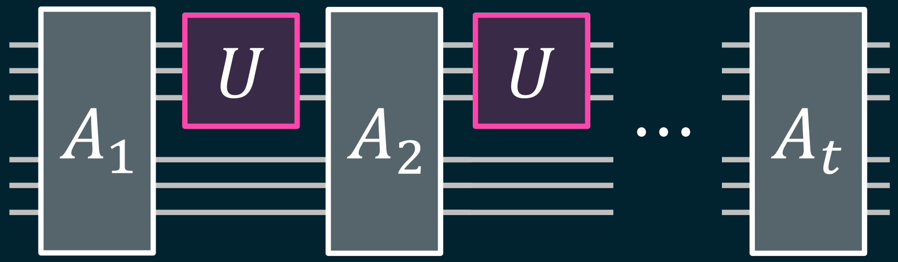

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- **Algorithms:** low-depth PRUs → faster quantum algorithms

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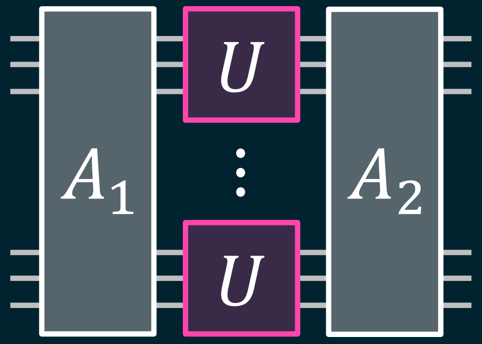

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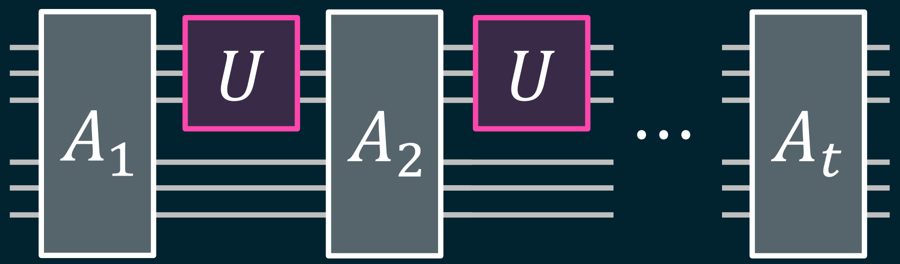

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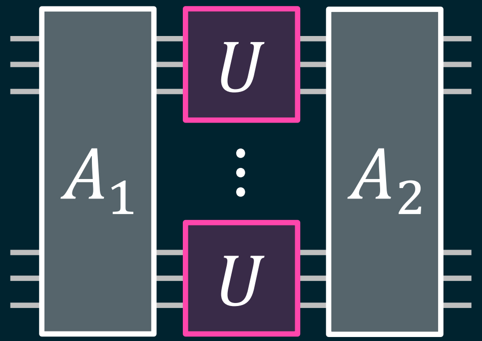

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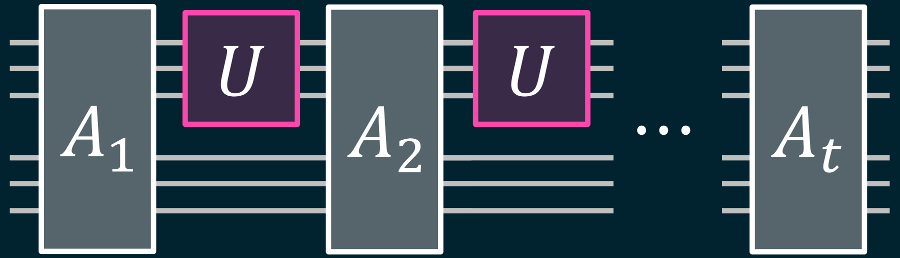

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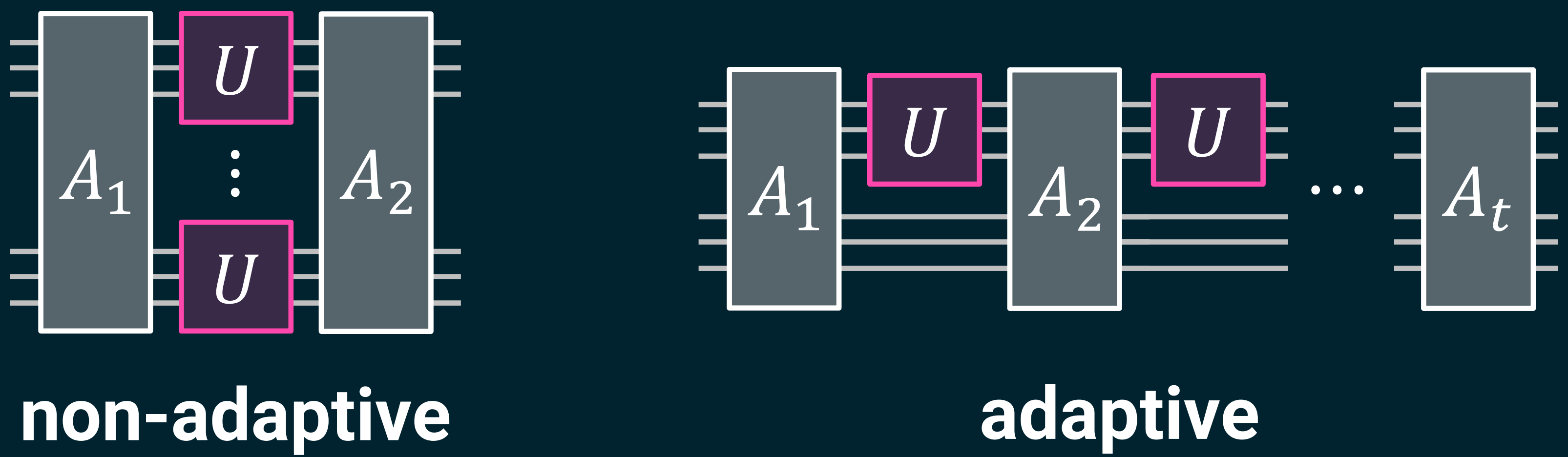

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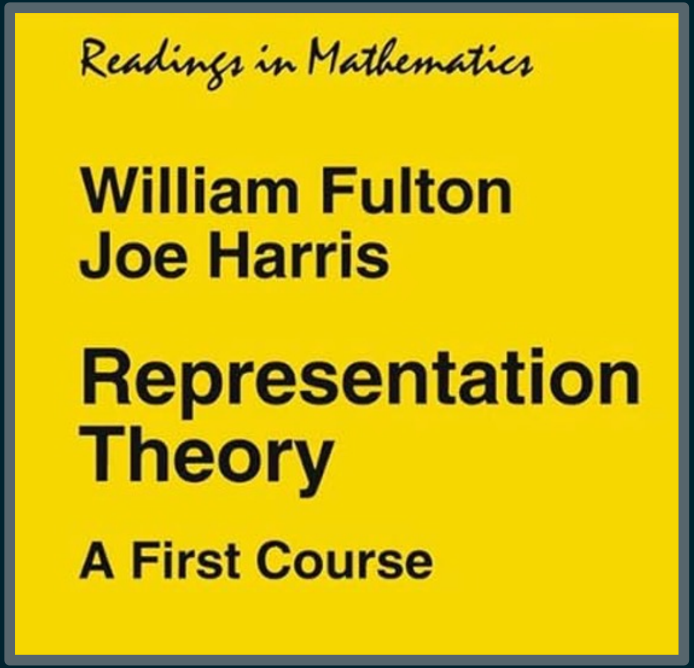

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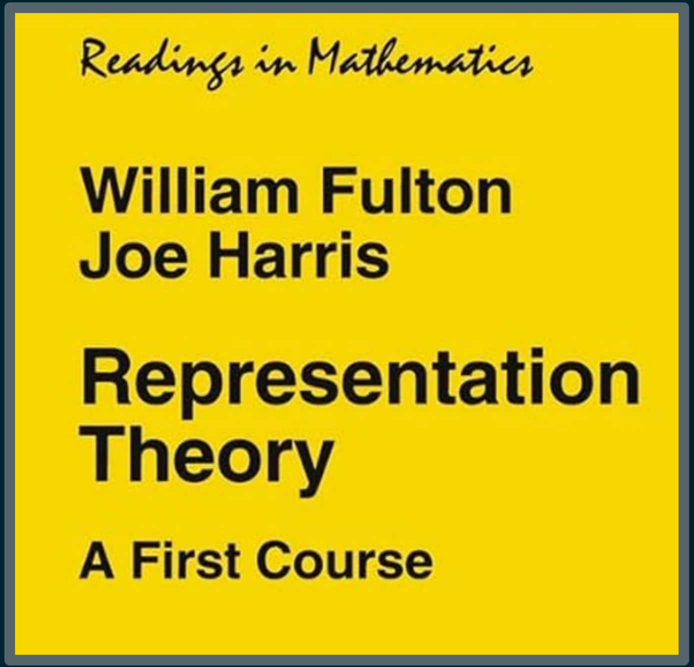

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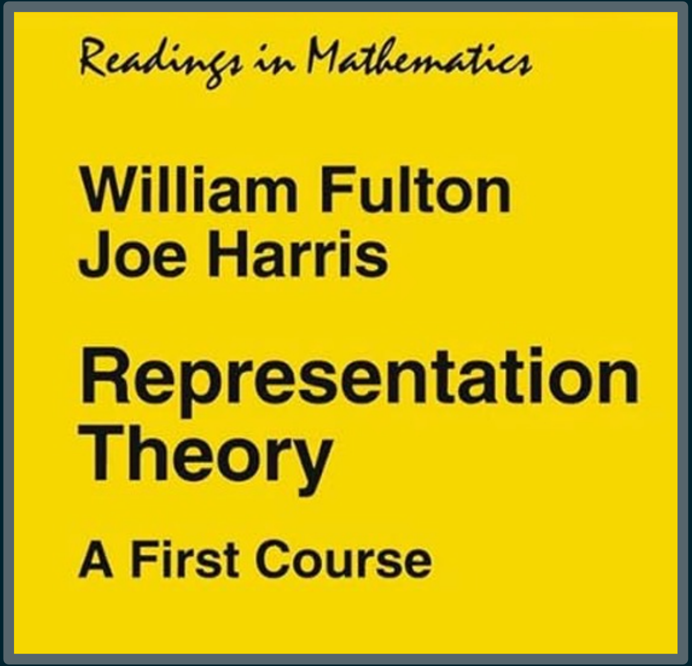

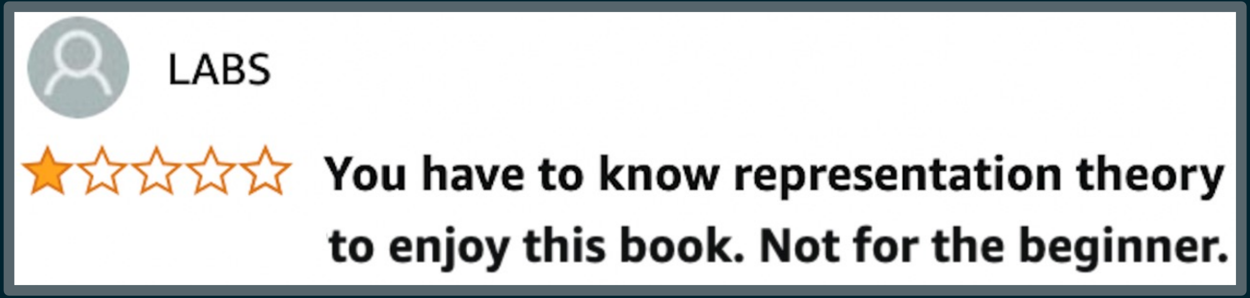

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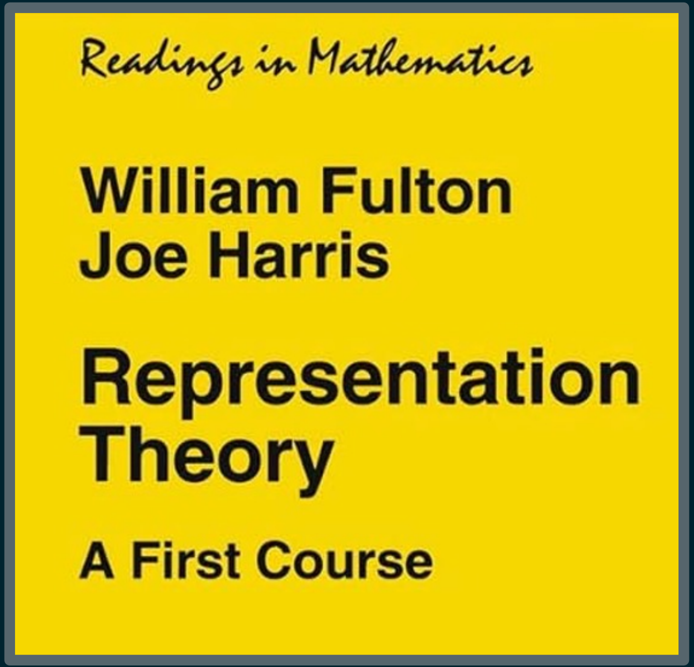

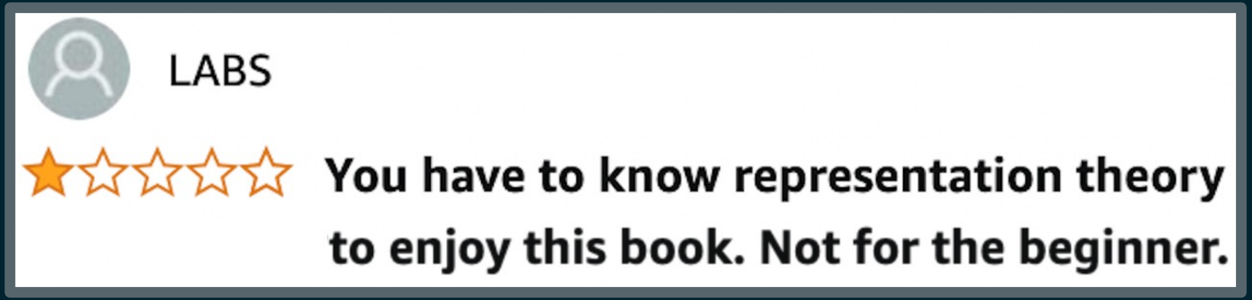

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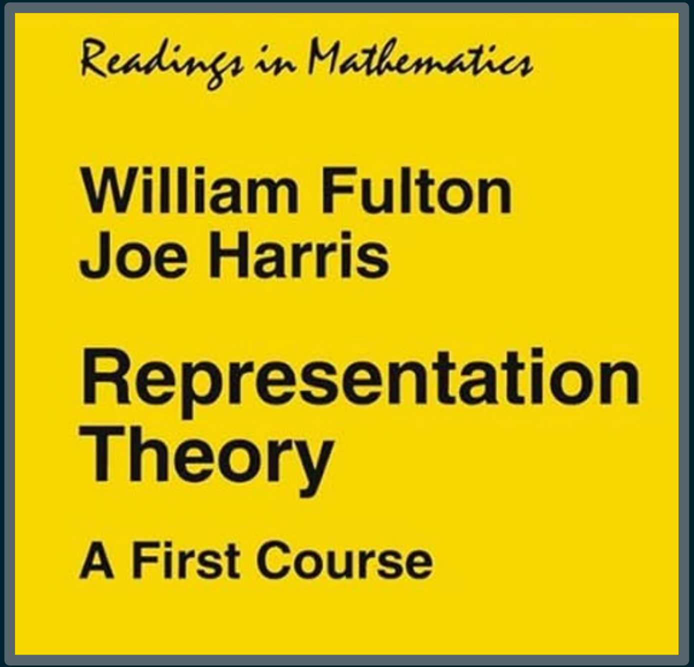

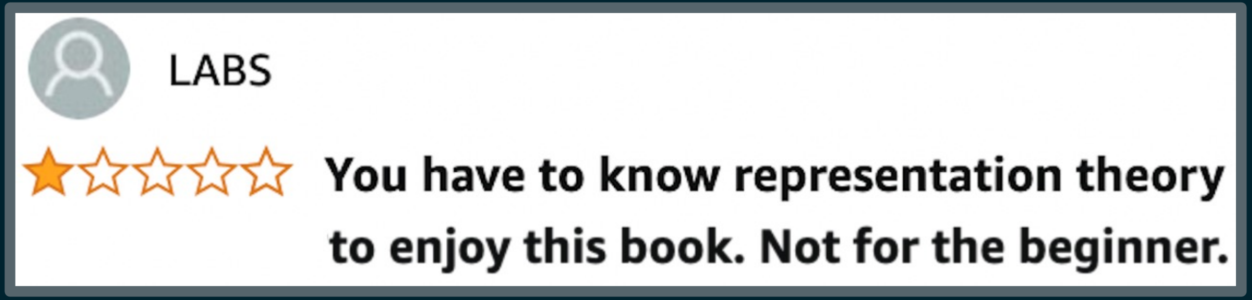

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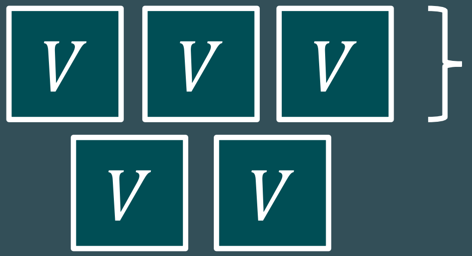
$U =$ 

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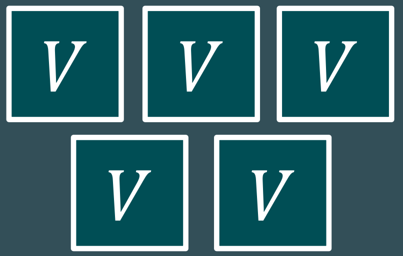

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$$\mathbb{E}_f\,|\psi_f\rangle\langle\psi_f|^{\otimes t} \approx \mathbb{E}_{U\leftarrow\mathrm{Haar}}\,\mathbb{E}_f\,(U|\psi_f\rangle\langle\psi_f|U^\dagger)^{\otimes t} = \mathbb{E}_{|\psi\rangle\leftarrow\mathrm{Haar}}\,|\psi\rangle\langle\psi|^{\otimes t}$$

(true for any U)

Recap of whiteboard proof

$$\sum_{f\in\{0,1\}^N} |\psi_f\rangle^{\otimes t}\,|f\rangle = \sum_{f\in\{0,1\}^N}\left(\sum_{x\in[N]}(-1)^{f\cdot e_x}|x\rangle\right)^{\otimes t}|f\rangle$$

$$\equiv \sum_{x_1,\dots,x_t\in[N]} |x_1,\dots,x_t\rangle\,|e_{x_1}\oplus\cdots\oplus e_{x_t}\rangle$$ (see whiteboard)

(after isometry) $$\approx \sum_{x_1,\dots,x_t\in[N]} |x_1,\dots,x_t\rangle\frac{1}{\sqrt{t!}}\sum_{\pi\in S_t} R_\pi|x_1,\dots,x_t\rangle$$ (see whiteboard)

For any unitary U: (by applying $\overline{U}^{\otimes t}$)

$$\sum_{x_1,\dots,x_t\in[N]} U^{\otimes t}|x_1,\dots,x_t\rangle\sum_{\pi\in S_t}R_\pi|x_1,\dots,x_t\rangle = \sum_{x_1,\dots,x_t\in[N]}|x_1\rangle\cdots|x_t\rangle\sum_{\pi\in R_\pi}R_\pi\cdot\cancel{U^{\top,\otimes t}}|x_1,\dots,x_t\rangle$$

using $\sum_{x\in[N]} U|x\rangle|x\rangle = \sum_{x\in[N]}|x\rangle\,U^\top|x\rangle$

This implies:

$$\mathbb{E}_f\,|\psi_f\rangle\langle\psi_f|^{\otimes t} \approx \mathbb{E}_{U\leftarrow\text{Haar}}\,\mathbb{E}_f\,(U|\psi_f\rangle\langle\psi_f|U^\dagger)^{\otimes t} = \mathbb{E}_{|\psi\rangle\leftarrow\text{Haar}}\,|\psi\rangle\langle\psi|^{\otimes t}$$

(true for any U) (invariance of Haar measure)