A one-query lower bound for unitary synthesis and breaking quantum cryptography

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(Simons and Berkeley)

joint work with Alex Lombardi and John Wright

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- **Factoring:** given a positive integer N, compute $f(N) =$ prime factorization of N.
- **Local Hamiltonian:** given a local Hamiltonian H, compute the bit $f(H)$ indicating whether H has a low-energy ground state.

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Physics: computing AdS/CFT map, decoding black-hole radiation

What can complexity theory say about the hardness of these **inherently quantum** problems?

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Before we continue:

1-minute detour for quantum computing 101

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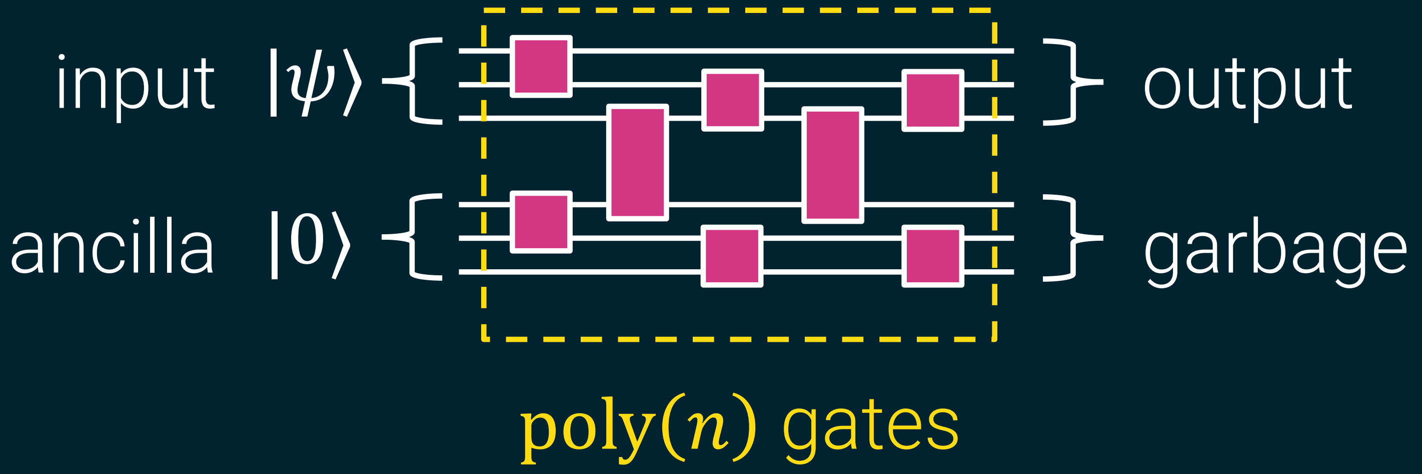

Now back to:

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To apply complexity theory, we need to ***efficiently reduce*** the task of implementing a unitary U to implementing a function f.

The Unitary Synthesis Problem [AK06]:

Is there a reduction for every unitary U?

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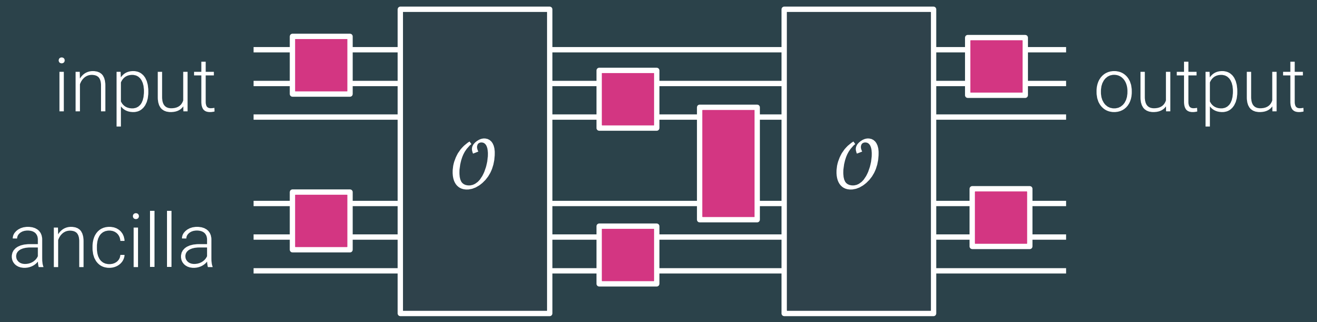

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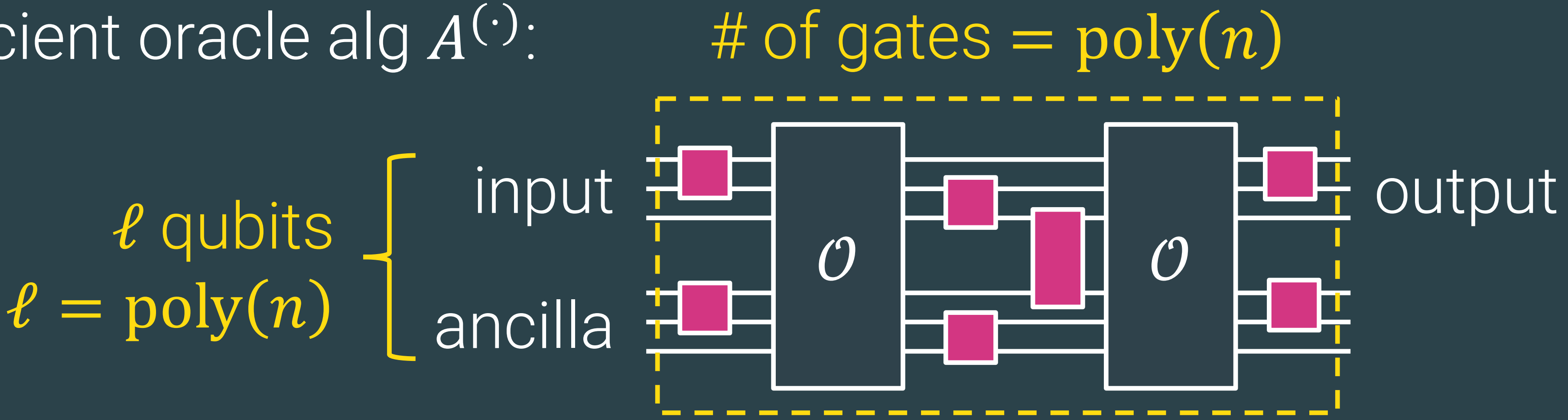

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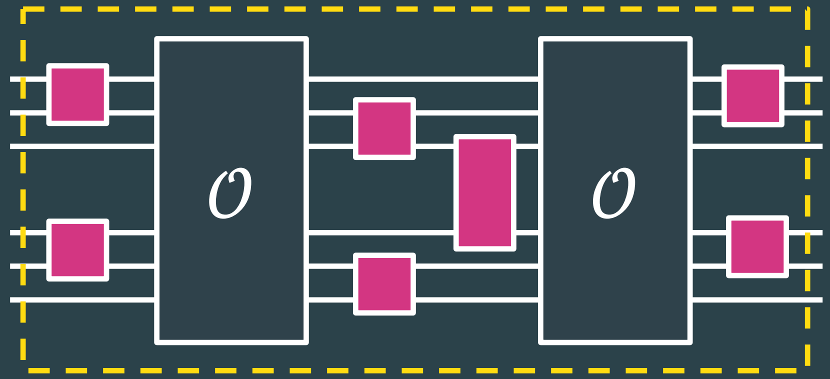

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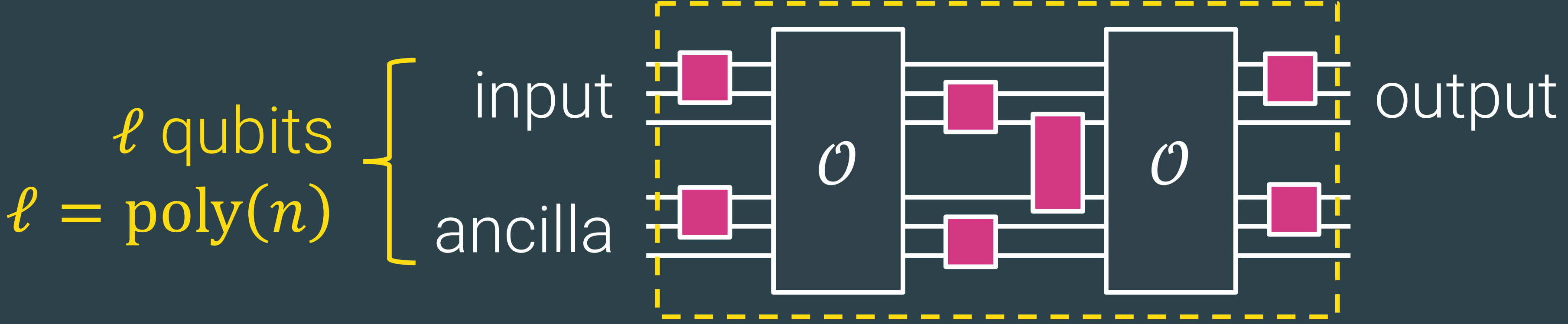

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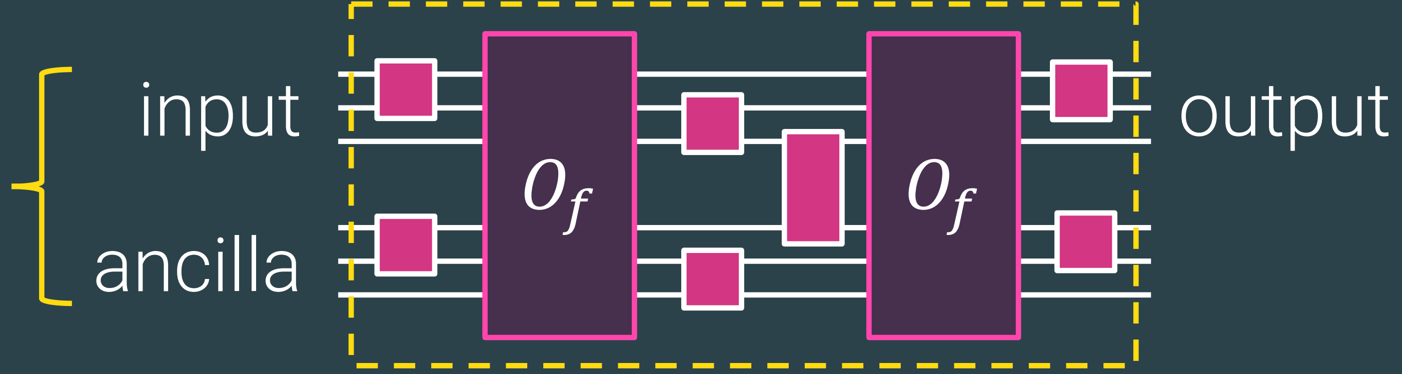

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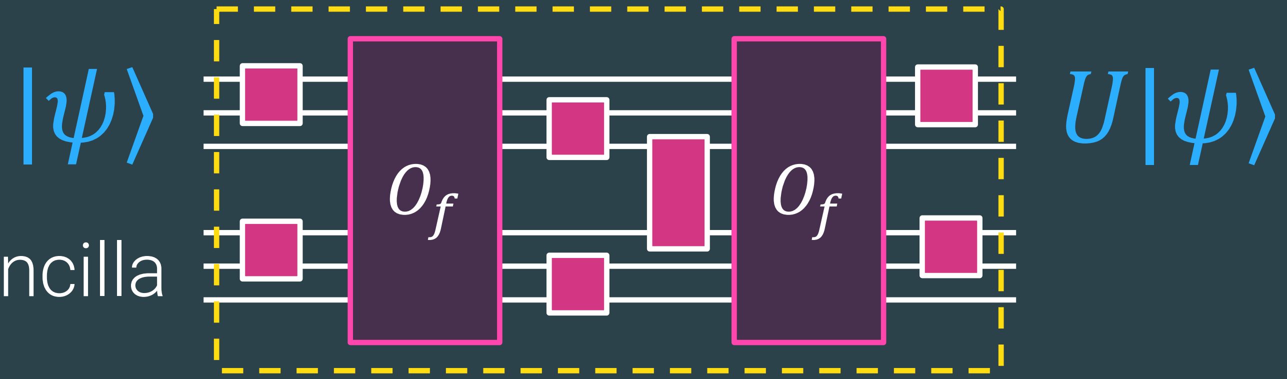

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Note: [AK06] prove a 1-query lower bound for a very special class of oracle algorithms.

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(2) Even one-query algorithms are very powerful!

In fact, they can solve any **classical input**, **quantum output** problem.

[Aar16, INNRY22, Ros23]

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Note: when $\ell = 2^{2n}$, possible to learn description of U in one query.

Rest of this talk

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Our answer: possibly harder than computing any function!

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Note: this result implies our unitary synthesis lower bound.

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given one query to a function f, which can depend on $R := \{R_k\}$.

Next up: what does a one-query adversary look like?

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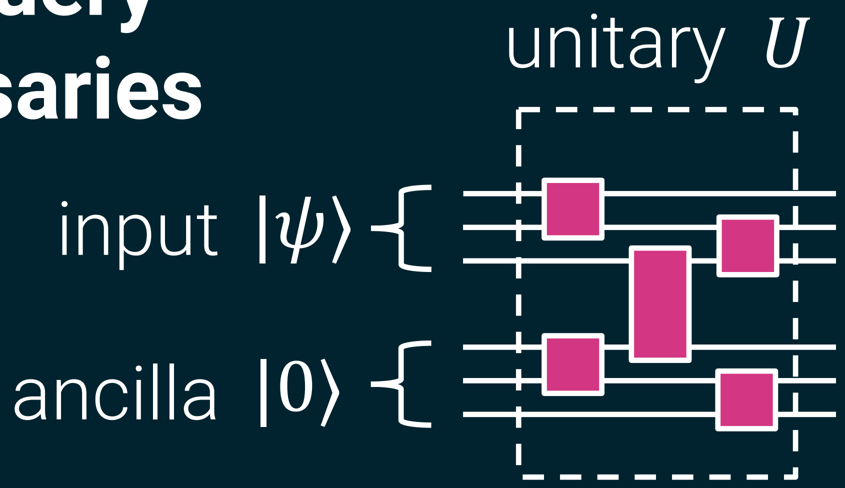

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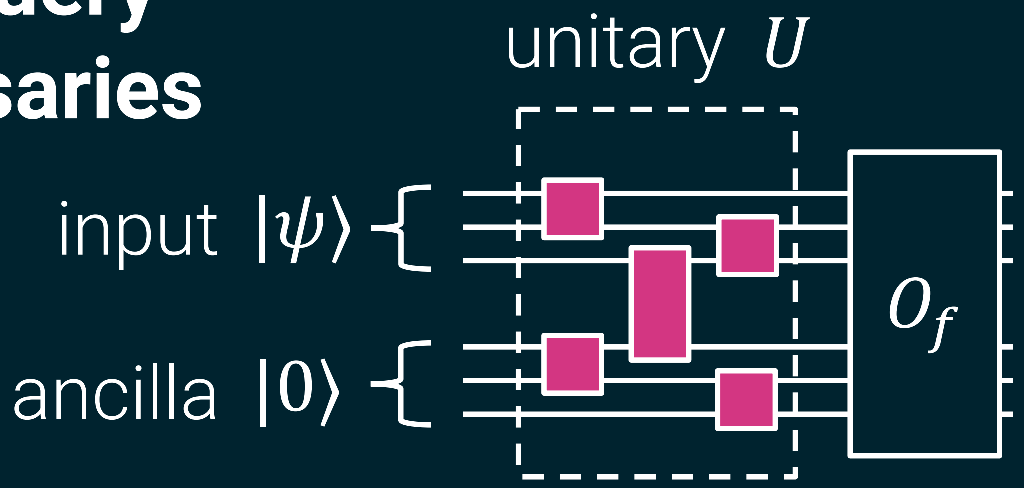

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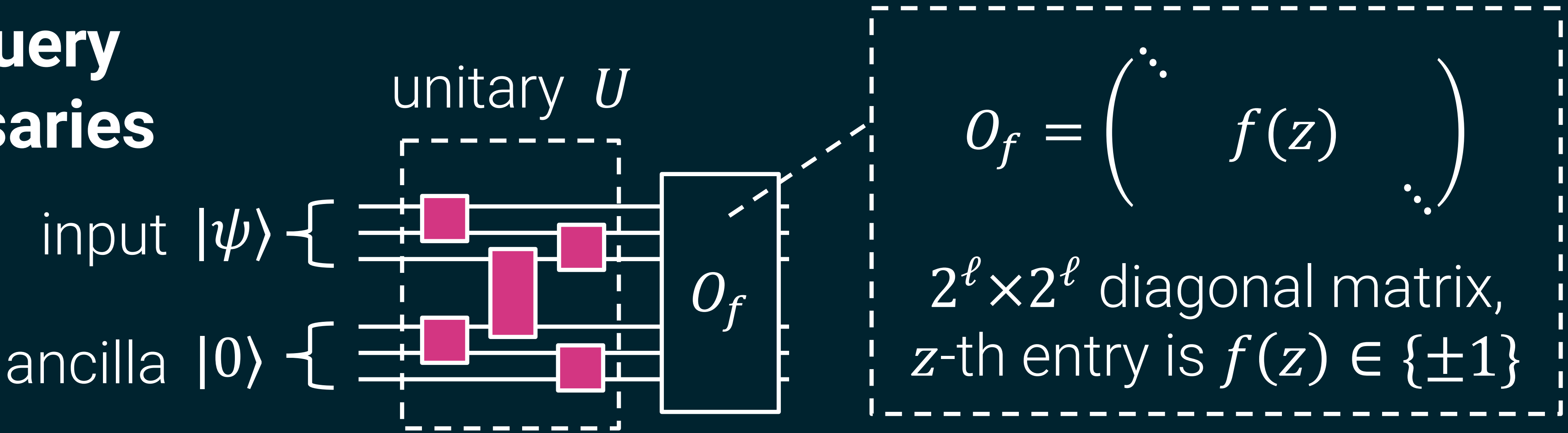

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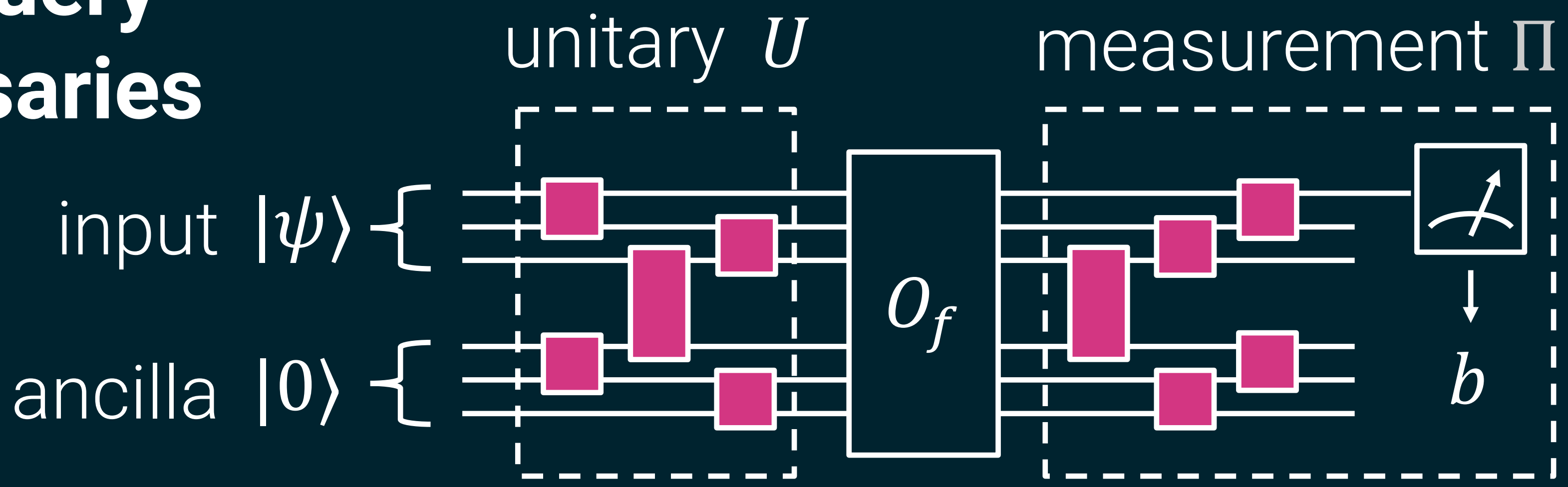

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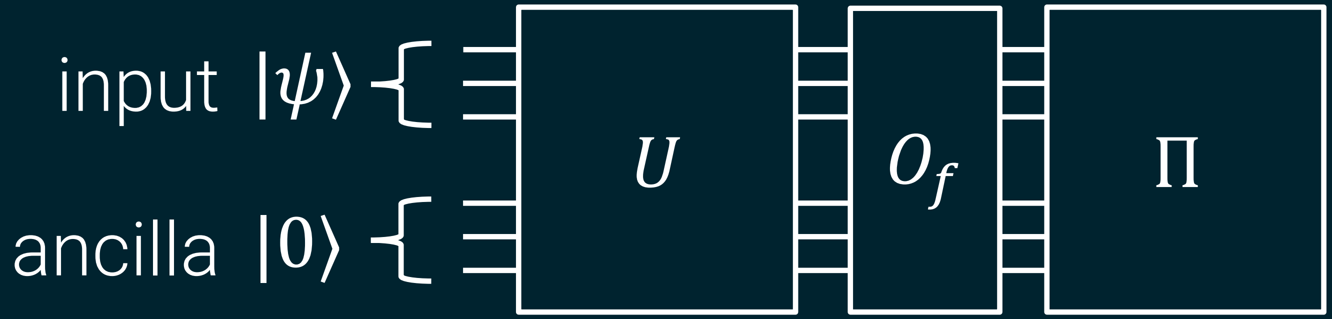

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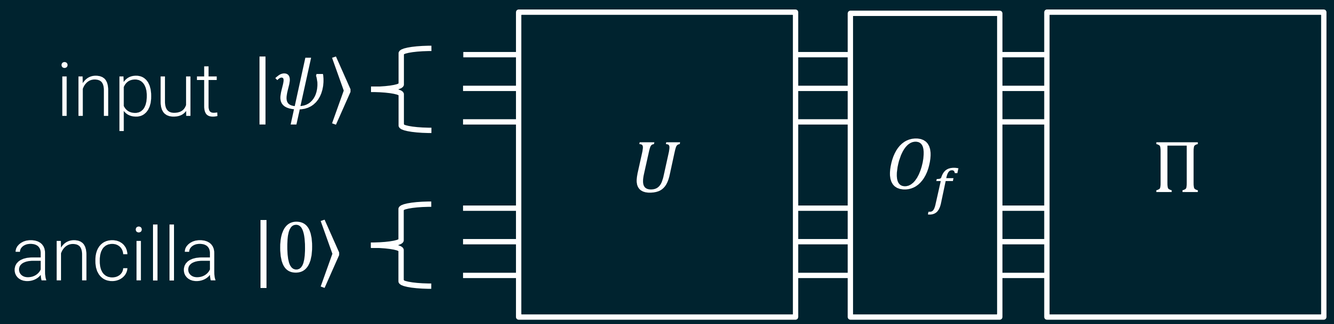

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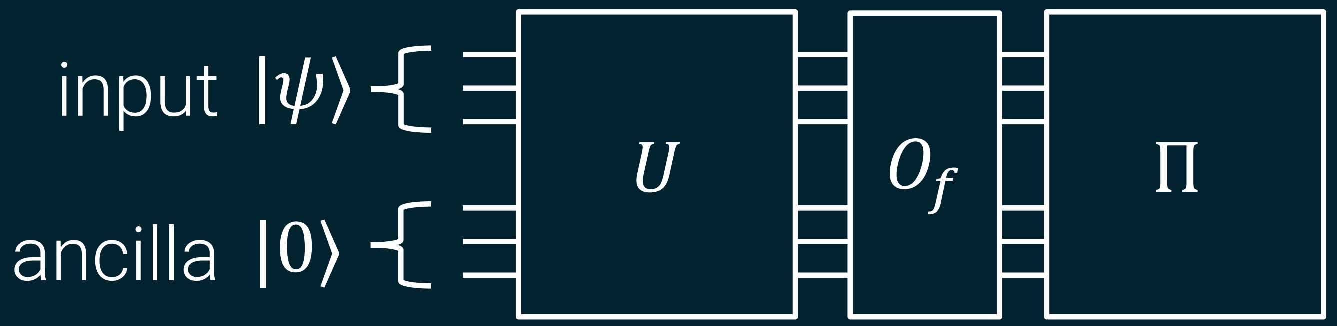

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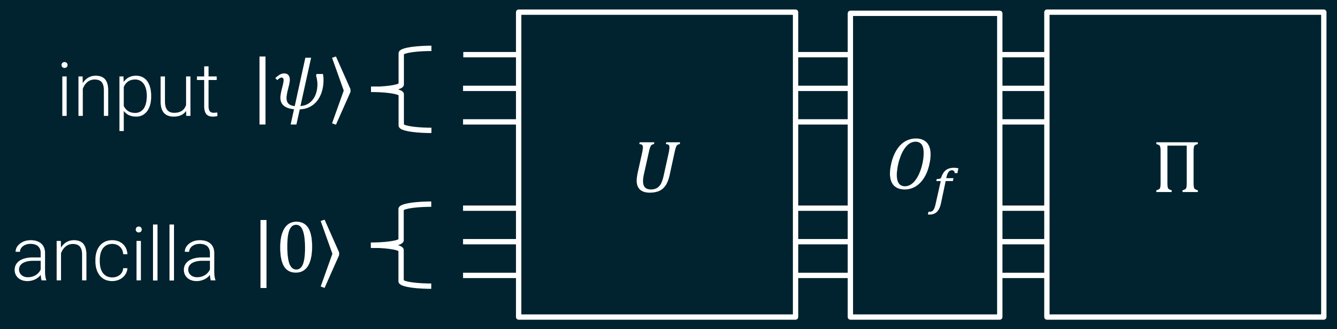

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1) Use spectral relaxation to bound distinguishing advantage in terms of the norm of a random matrix

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(adversary picks $f = f_R$ to maximize this)

Goal: bound **maximum distinguishing advantage**.

The plan:

1) Use spectral relaxation to bound distinguishing advantage in terms of the norm of a random matrix

2) Apply matrix concentration

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Rest of this talk

Part 1:
Connect unitary synthesis to breaking quantum cryptography

Part 2:
A special case of our proof

A special class of one-query adversaries

Assume adversary sets $\ell = n$ (no ancillas) and $U = \mathrm{Id}$.

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Disclaimer: We can rule out these attacks with a counting argument, but today we'll see a different proof.

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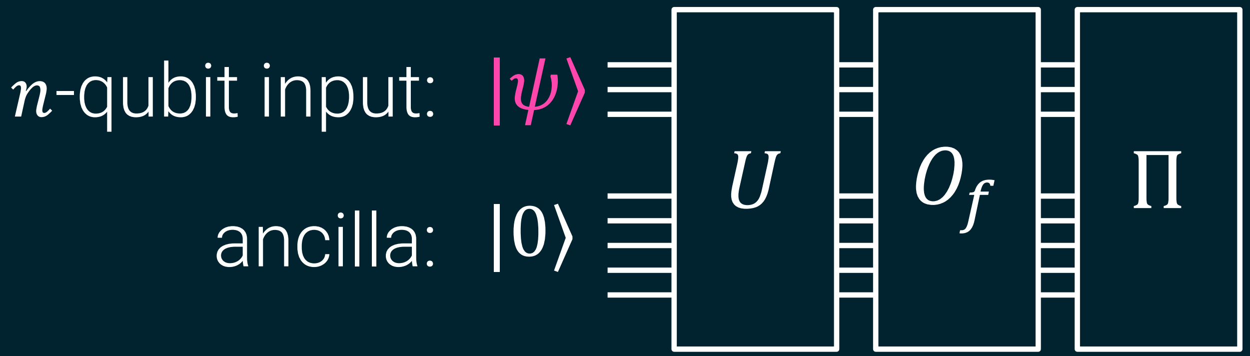

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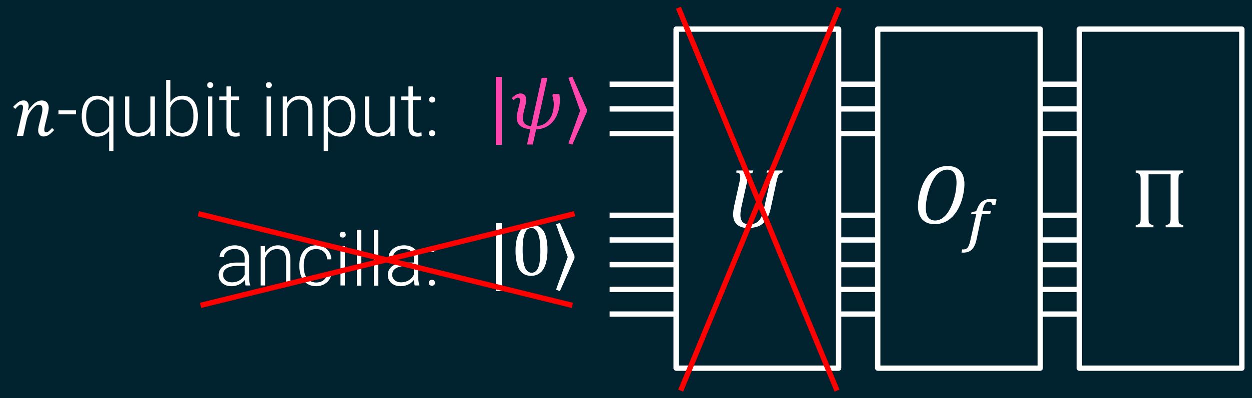

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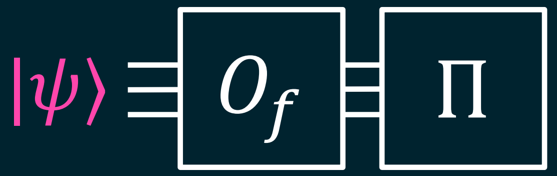

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Adversary's advantage (for this special class):

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$\Downarrow$?

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Then matrix Chernoff will bound the max over **all unit vectors**.

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Since all the terms look identical, it suffices to just look at one term.

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$N \times N$ diagonal matrix,
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Distinguishing advantage

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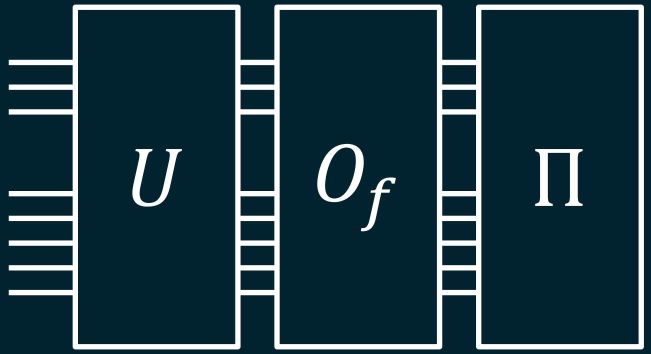
by Matrix Chernoff with $X_k = D_{R_k}\cdot\Pi\cdot D_{R_k}$

How do we handle general one-query adversaries?

General one-query adversaries

n qubit input: $|\psi_h\rangle$

ancilla: $|0\rangle$



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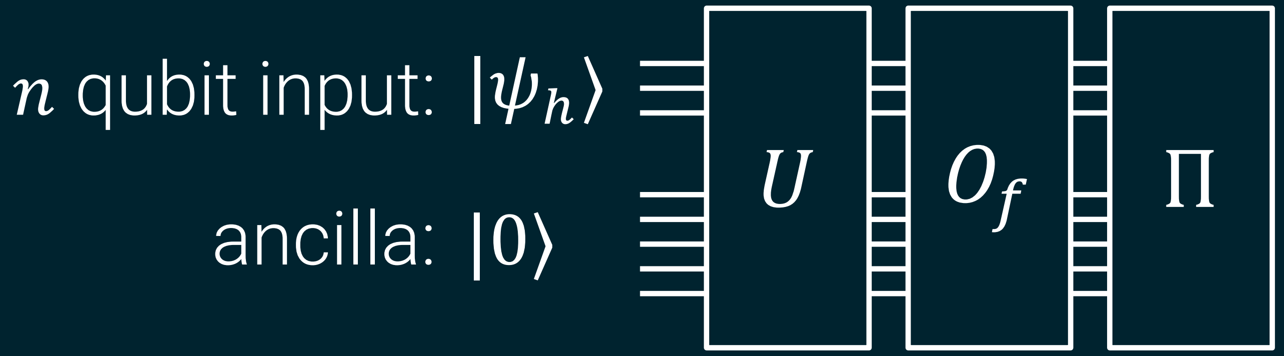

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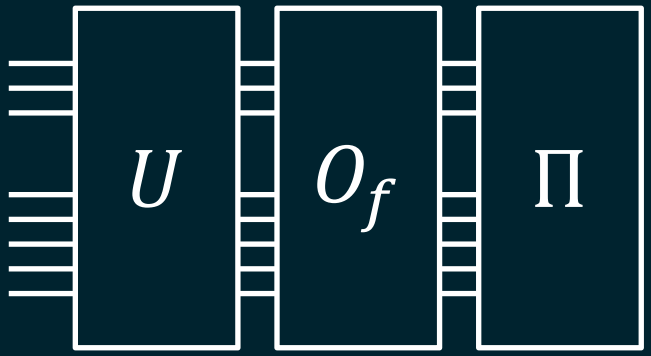

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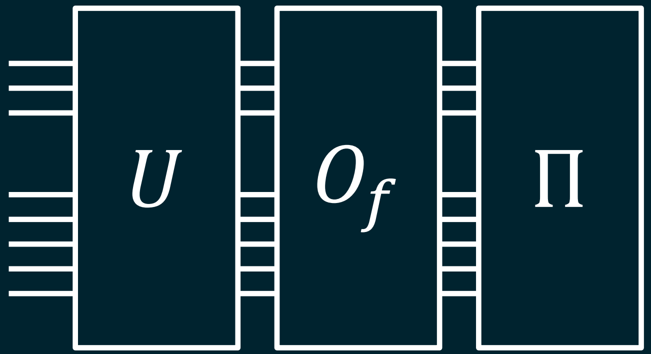

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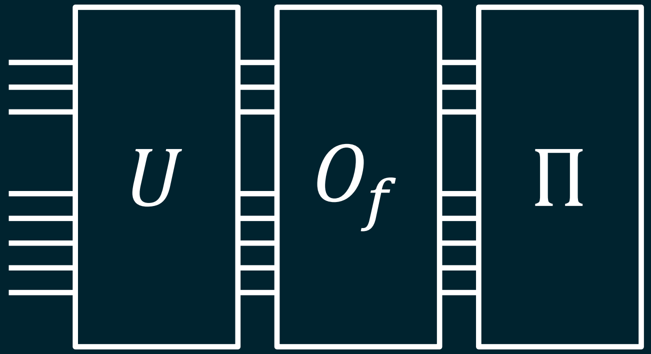

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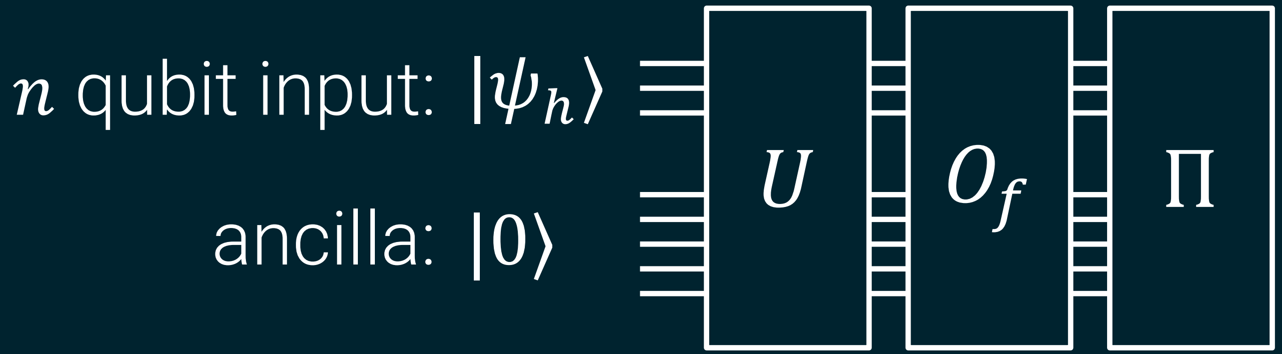

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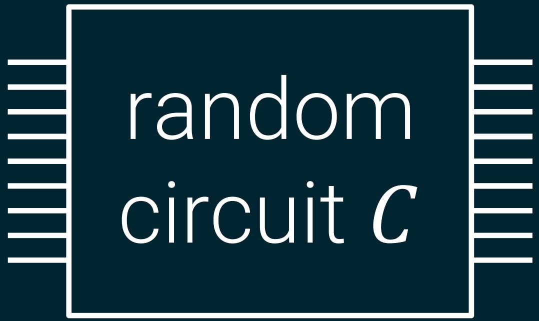

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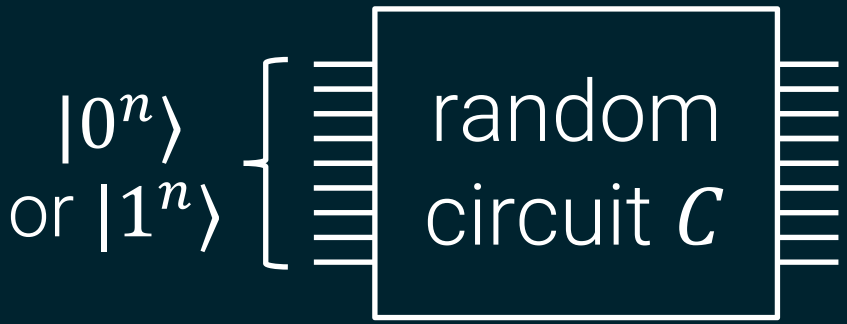

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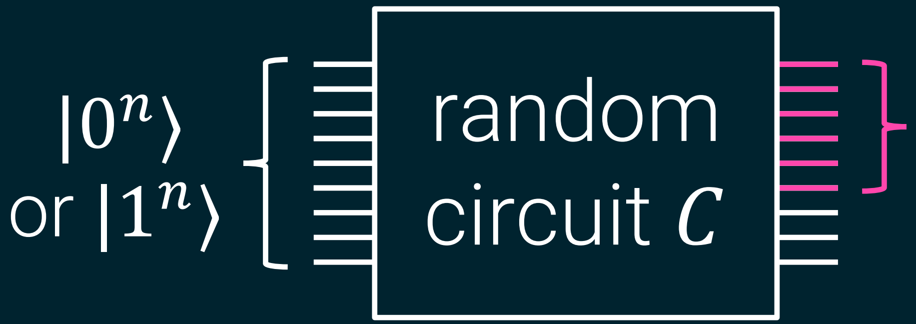

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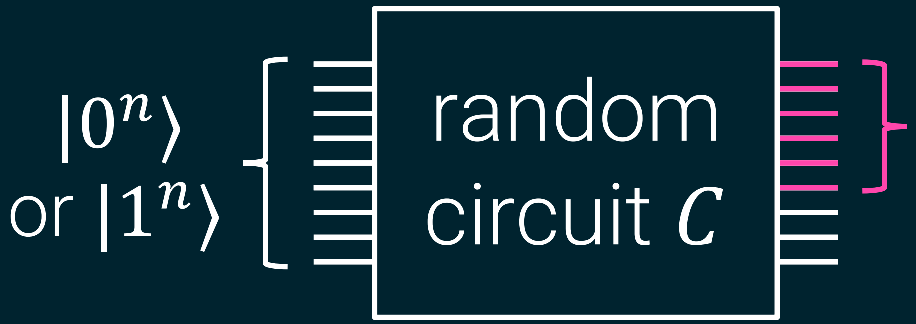

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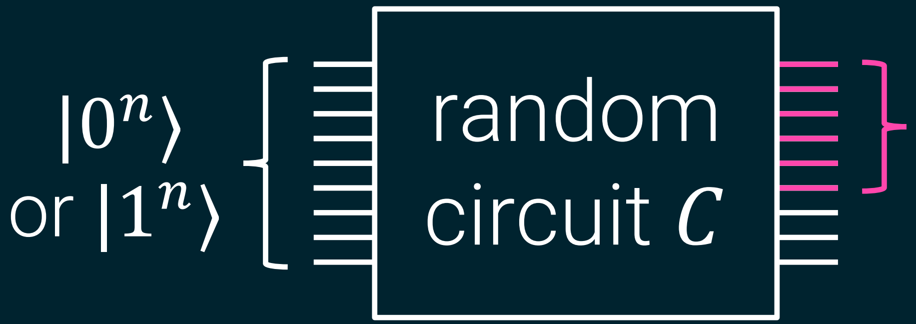

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Thanks for listening!