A one-query lower bound for unitary synthesis and breaking quantum cryptography

Fermi Ma

(Simons and Berkeley)

joint work with Alex Lombardi and John Wright

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Key point: even though this problem is "about" quantum states, the input and output are classical.

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This talk:

How hard are “inherently quantum” problems?

Before we continue:

1-minute detour for quantum computing 101

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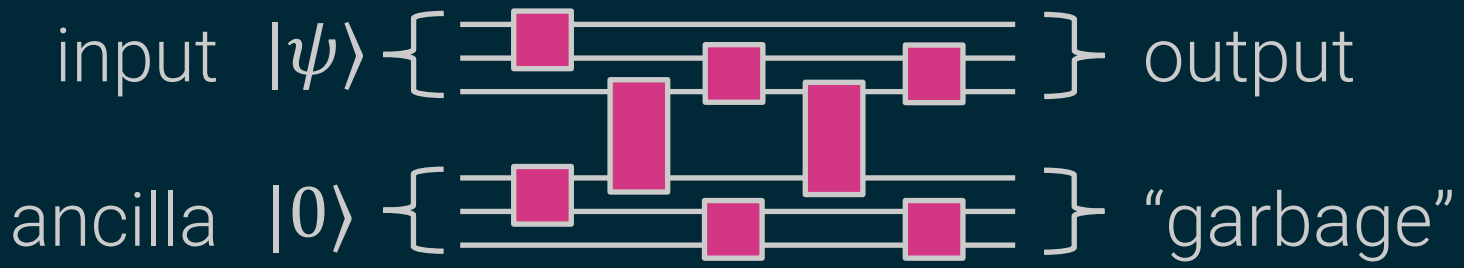

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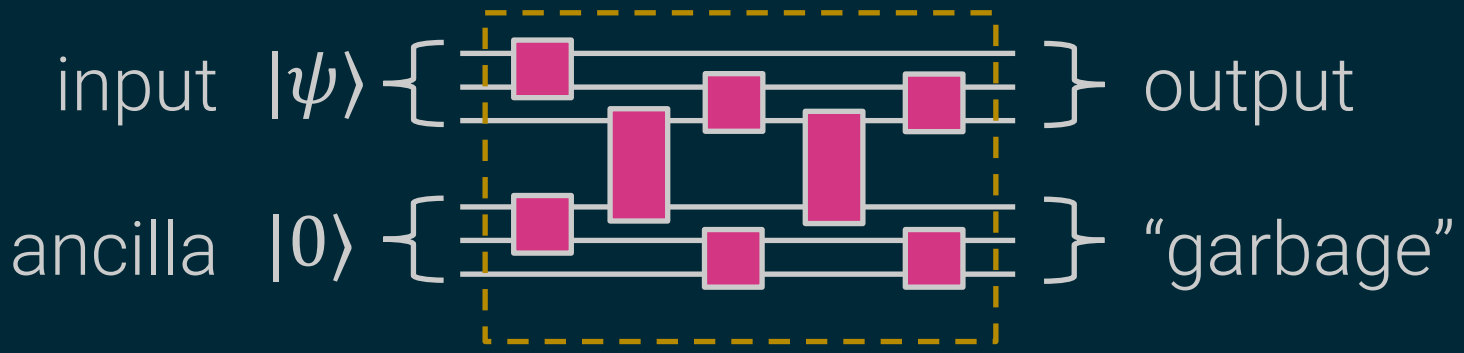

efficient unitary $= \mathbf{poly}(n)$-size circuit

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[INNRY22]: Search-QMA is easy given one query to a $\mathbf{PP}$ oracle.

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Not obvious: Is there ***any*** oracle that makes this easy? Even an oracle for the halting problem?

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This is the Unitary Synthesis Problem [AK06].

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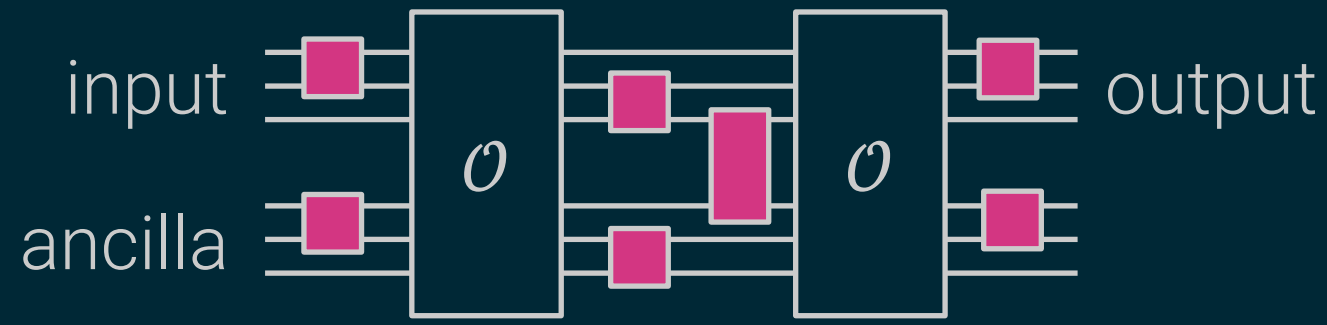

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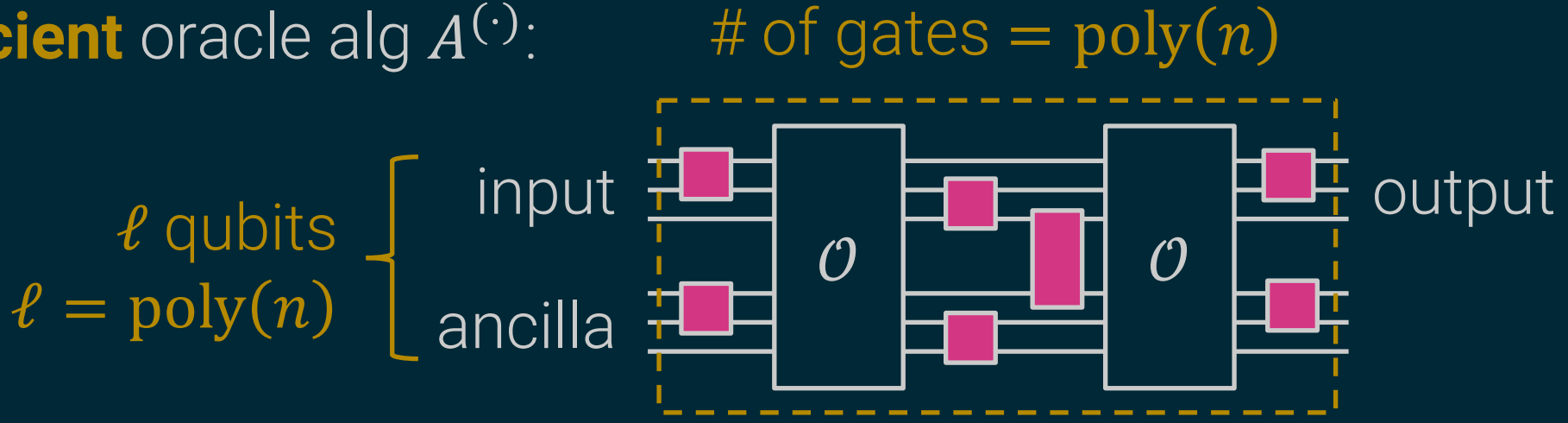

# of gates $= \mathrm{poly}(n)$

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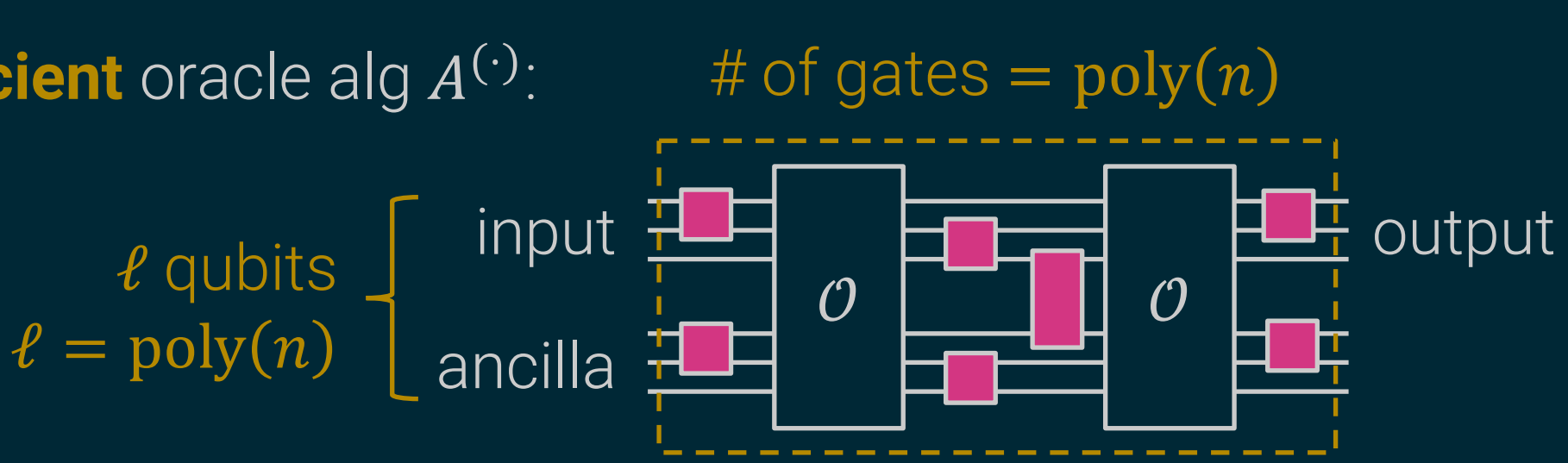

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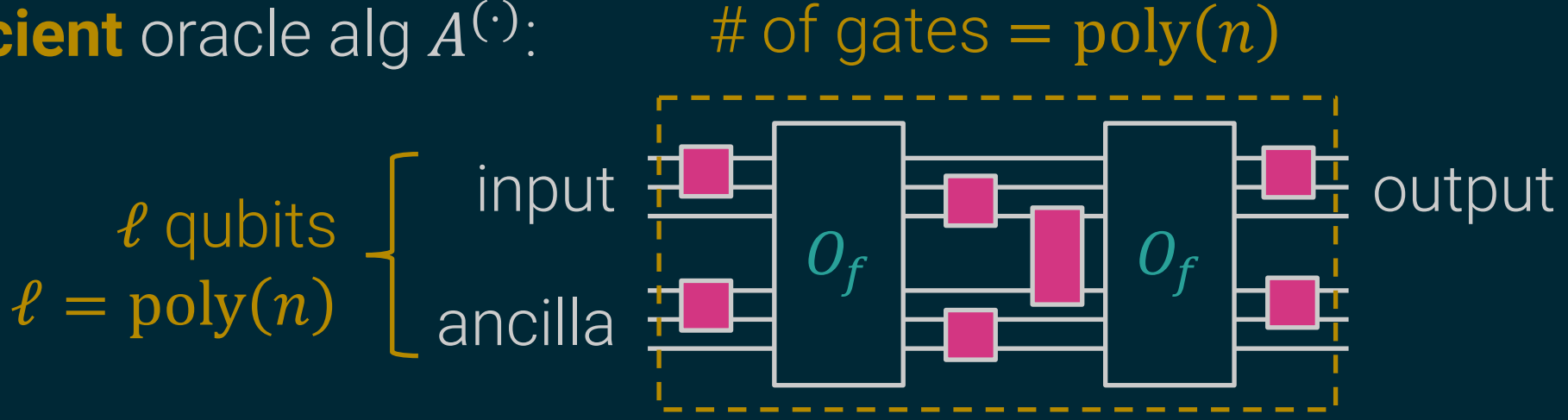

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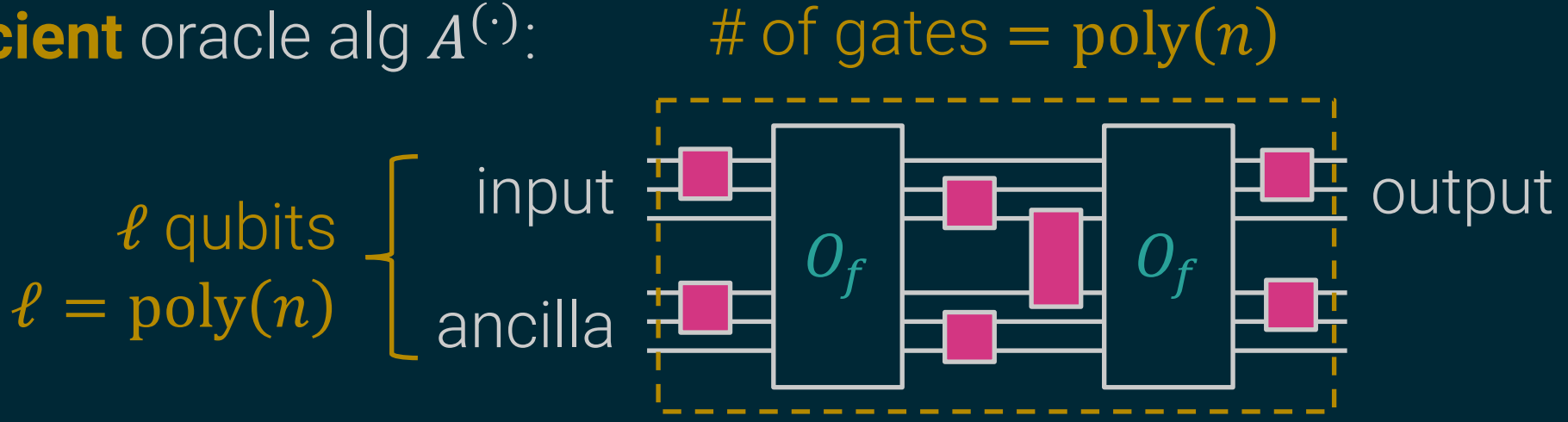

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Rephrased: can quantum problems (unitaries) be **efficiently reduced** to classical problems (functions)?

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Lower bounds

- [AK06]: 1-query lower bound for very restricted class of algorithms.

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(2) Even ***one-query*** algorithms are very powerful!

In fact, they can solve any **classical input**, **quantum output** problem.
[Aar16, INNRY22, Ros23]

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In fact, we rule out oracle algorithms $A^{(\cdot)}$ with:

- unlimited space (number of qubits)
- unlimited size (number of quantum gates)
- one query to a function $f:\{0,1\}^{\ell} \to \{\pm 1\}$ where $\ell = o(2^n)$. (Bernstein-Vazirani algorithm requires $\ell = 2^{2n}$)

Plan for this talk

- Relating unitary synthesis to breaking quantum cryptography
- The Oracle State Distinguishing Game
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Next: brief crypto background + explain this connection

Cryptographic pseudorandomness

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If $\mathrm{P} = \mathrm{NP}$, all PRGs are broken:

- Given any $\mathbf{NP}$-complete $f\colon\{0,1\}^* \to \{\pm 1\}$, there is a poly-time oracle algorithm A^f that breaks any PRG (Turing reduction).

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Indicates that the existence of quantum crypto could be independent of traditional complexity theory.

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(e.g., if PRS states $\{|\psi_k\rangle\}_{k\in[K]}$ are orthogonal, the unitary that maps $|\psi_k\rangle \to |k\rangle$ would break the PRS.)

Main result #1: There is no efficient one-query oracle algorithm $A^{(\cdot)}$ for the Unitary Synthesis Problem.

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- Relating unitary synthesis to breaking quantum cryptography
- **The Oracle State Distinguishing Game**
- If time: our proof for a special class of adversaries

Our approach: analyze the task of breaking an n-qubit PRS where the $K \ll N \coloneqq 2^n$ states are random binary phase states.

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- A **function family** $R\colon [K] \times [N] \to \{\pm 1\}$ specifies K binary phase states $\{|\psi_{R_k}\rangle\}_{k \in [K]}$ where $R_k(x) := R(k, x)$.

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Our lower bound holds up to $\ell = o(2^n)$.

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Game: Sample $b \leftarrow \{0,1\}$.

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1-query adversaries: Adversary makes one query to an arbitrary function $f_R\colon \{0,1\}^{\ell} \to \{\pm 1\}$ that depends on R.

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Adversary wins $b = b'$.

One-query adversaries

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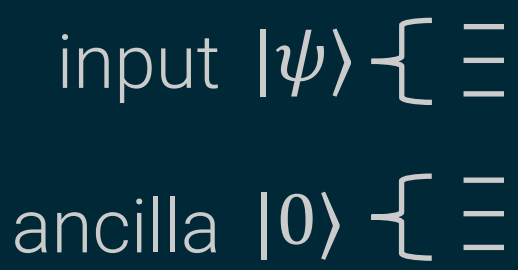

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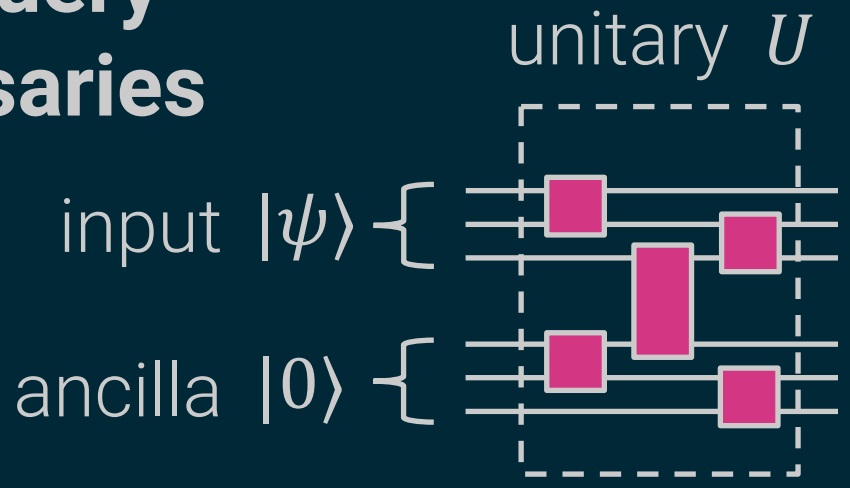

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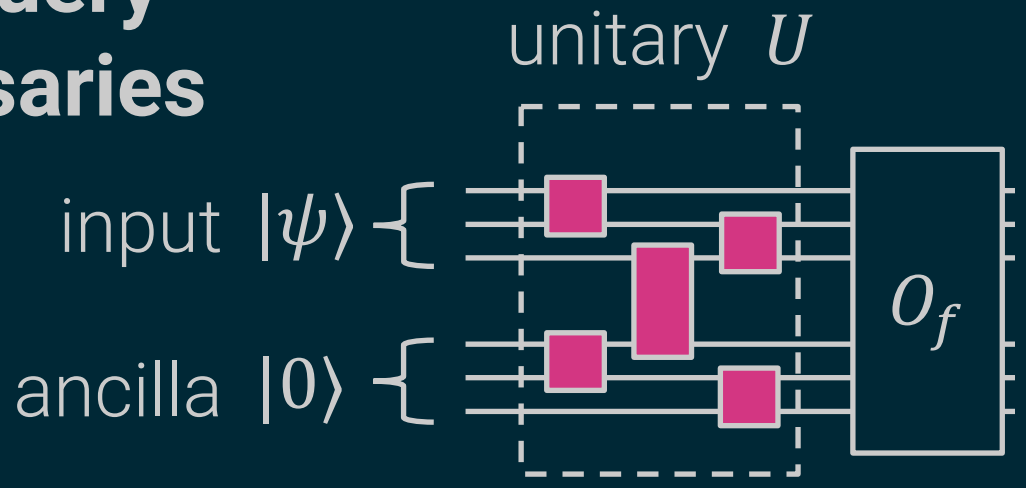

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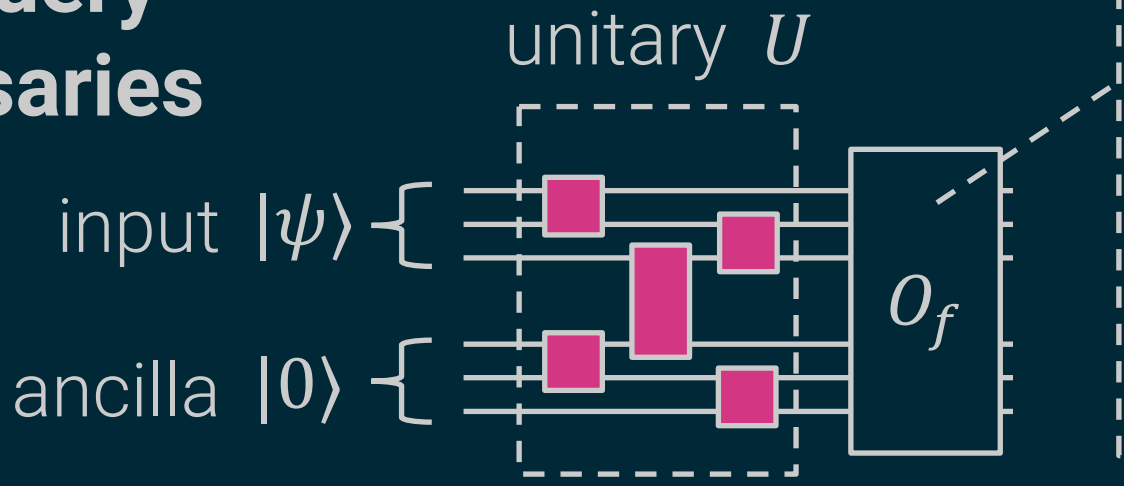

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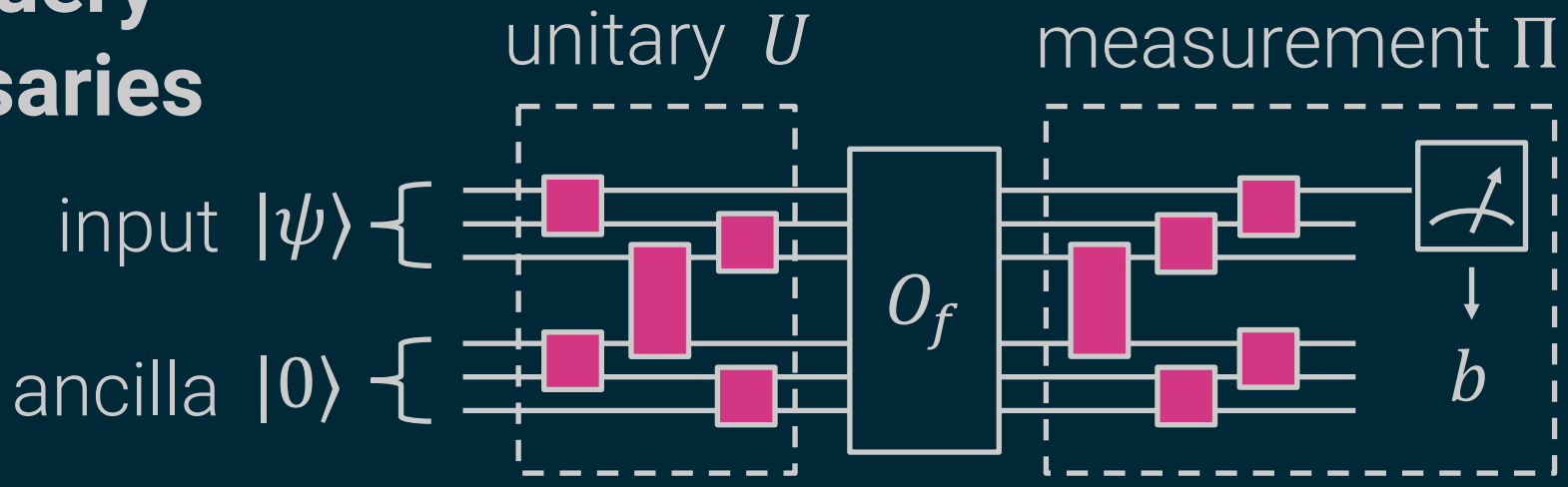

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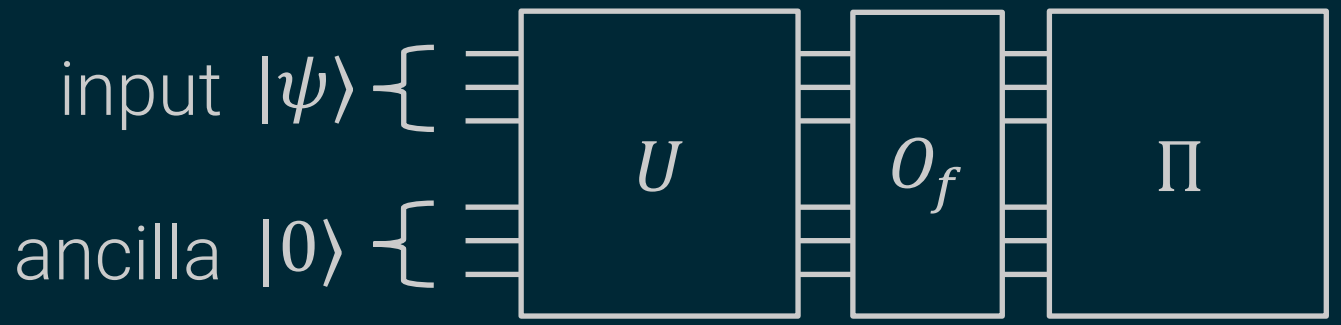

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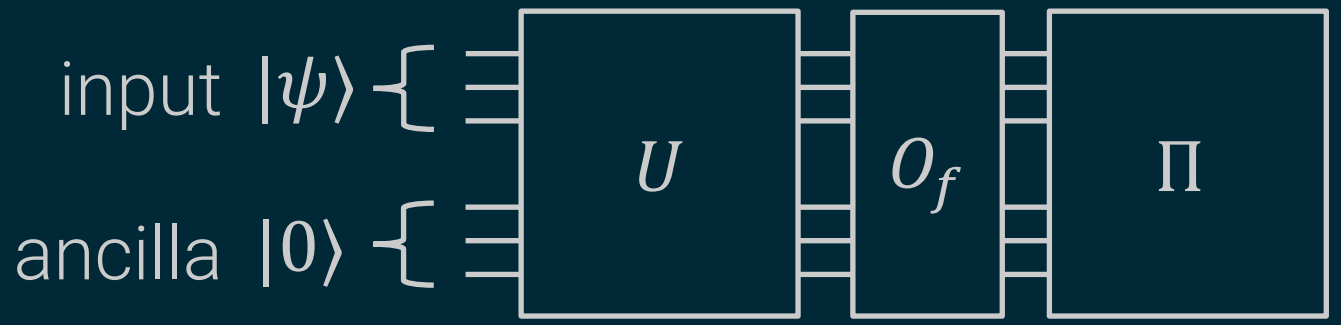

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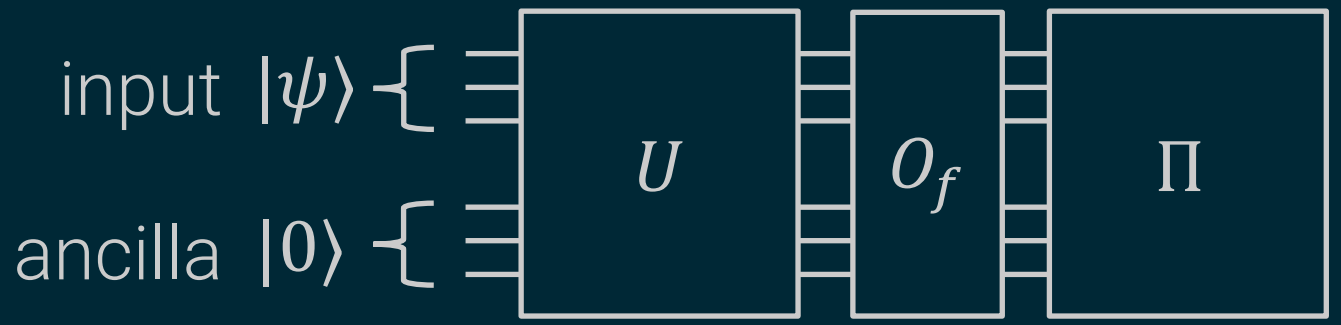

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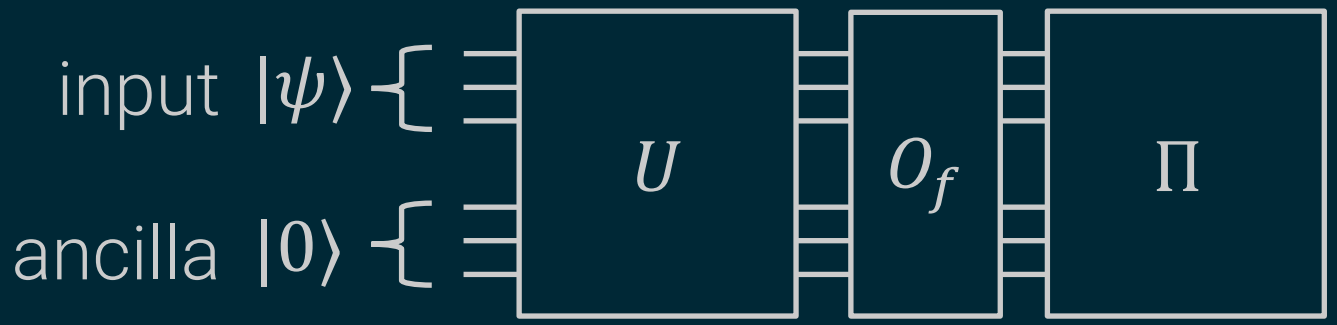

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We show:

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We show:

- Carefully-chosen **spectral relaxation** gives an upper bound in terms of the operator norm of a certain random matrix.
- We bound this norm by appealing to **matrix concentration**.

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Plan for this talk

- Relating unitary synthesis to breaking quantum cryptography
- The Oracle State Distinguishing Game
- **If time: our proof for a special class of adversaries**

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Assume adversary sets $\ell = n$ (no ancillas) and $U = \mathrm{Id}$.

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Disclaimer: We can rule out these attacks with a counting argument, but today we'll see a different proof.

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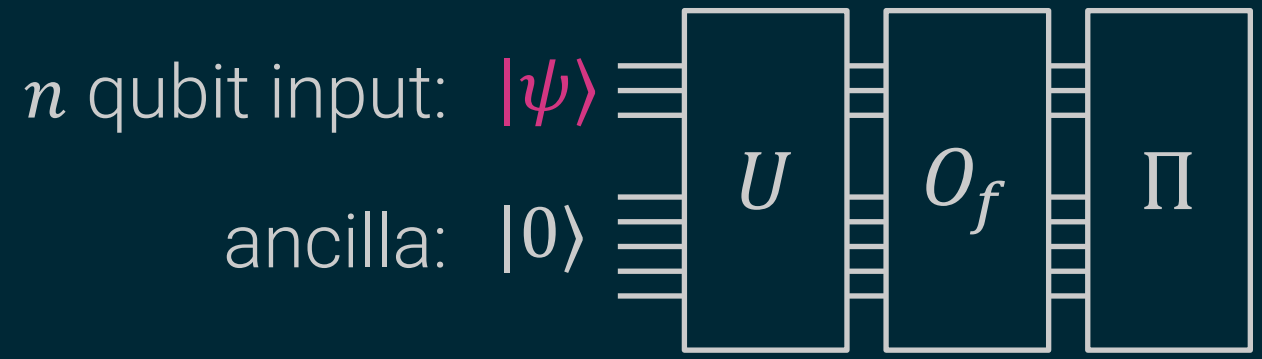

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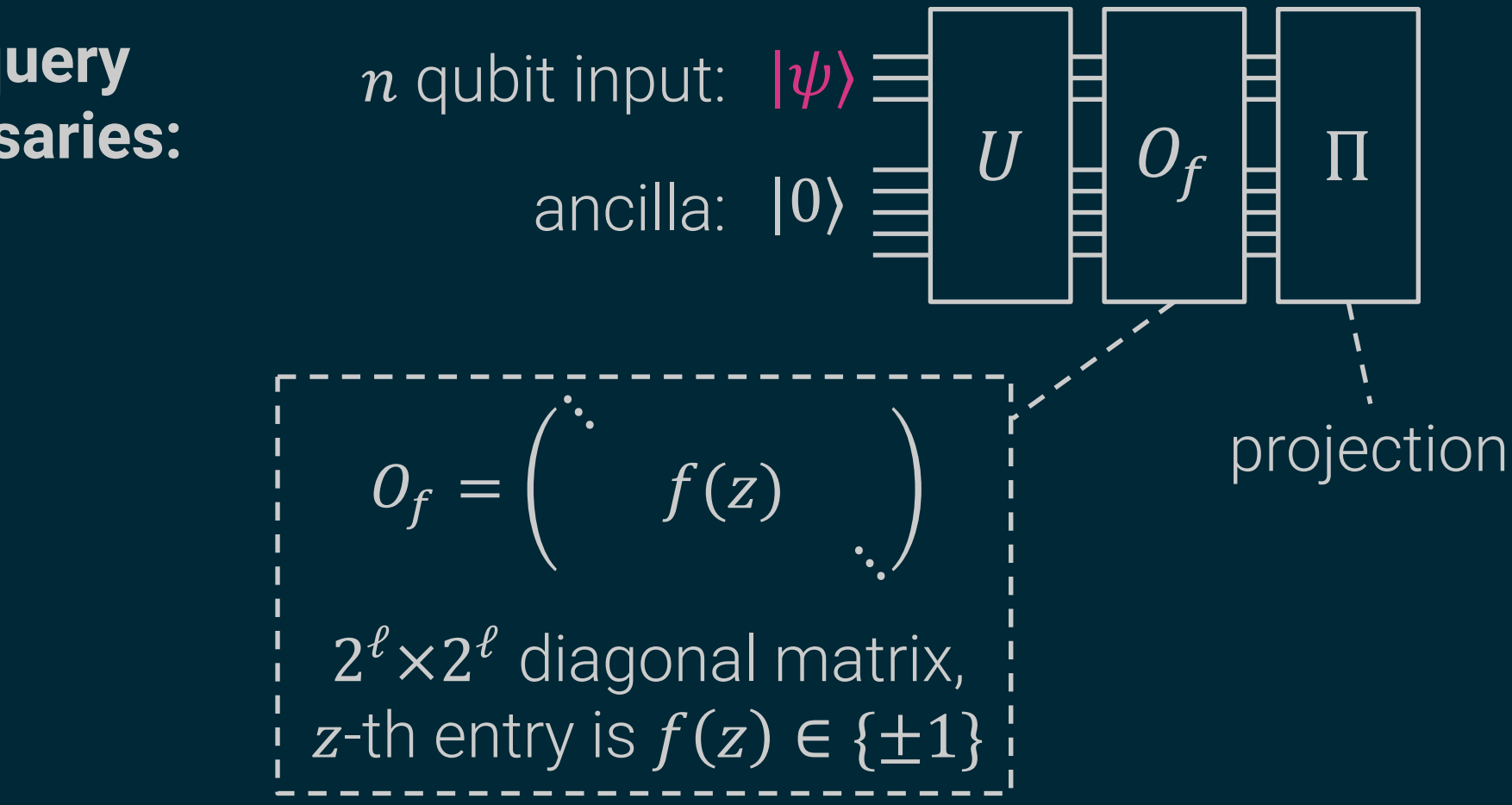

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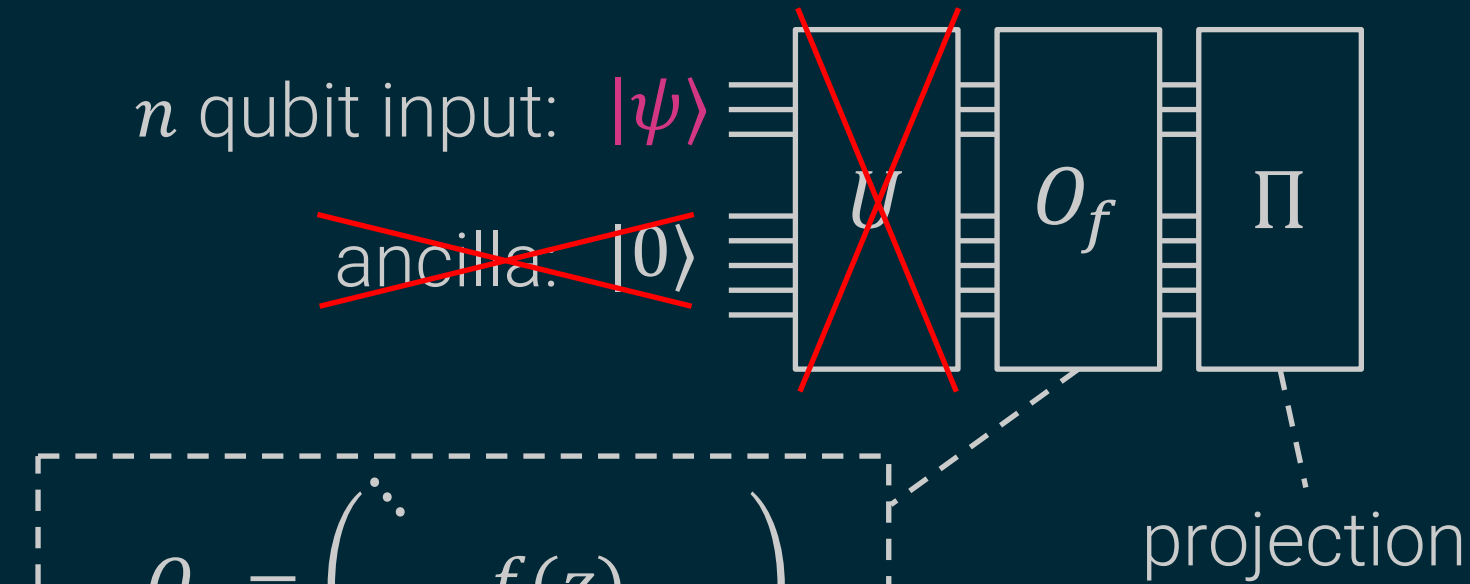

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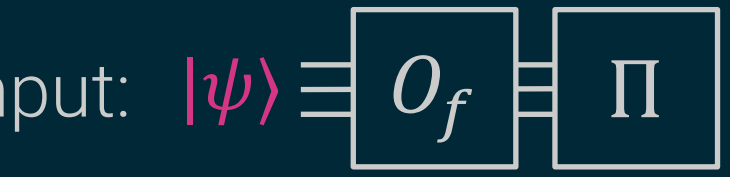
Special class: n qubit input: 

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Adversary's advantage (for this special class):

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Let's see how!

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(1) Write the binary phase state $|\psi_{R_k}\rangle$ as

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uniform superposition

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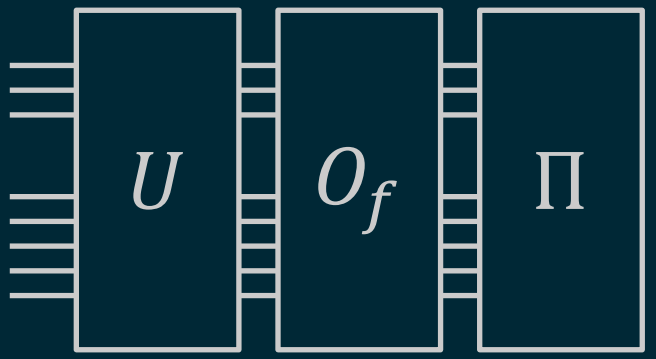
Extending this proof to general one-query adversaries requires more care.

General one-query adversaries

n qubit input: $|\psi_h\rangle$

ancilla: $|0\rangle$



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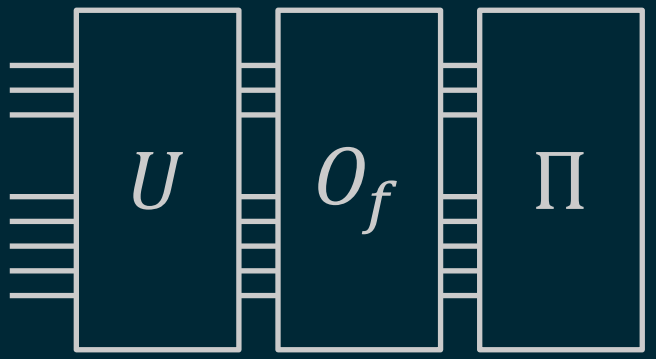

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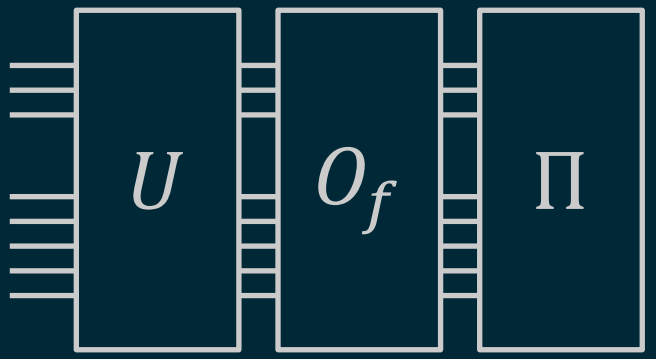

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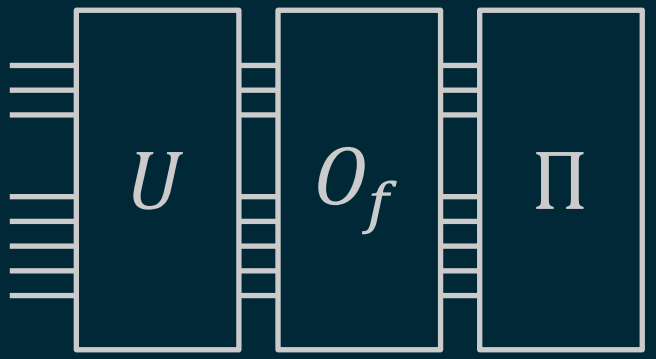

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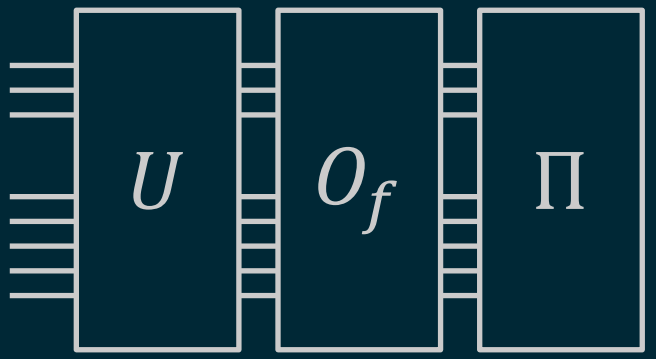

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Idea: obtain spectral relaxation by factoring $V|\psi_h\rangle = \widetilde{D_h} \cdot |\mathrm{wt}_V\rangle$ w.r.t. a V-dependent unit vector $|\mathrm{wt}_V\rangle$.

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Thanks for listening!