A one-query lower bound for unitary synthesis and breaking quantum cryptography

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(Simons and Berkeley)

joint work with Alex Lombardi and John Wright

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- **Hamiltonian cycle:** given a graph G, compute any function $f(G)$ whose output is a Hamiltonian cycle of G.
- **Local Hamiltonian:** given a local Hamiltonian H, output $f(H) \in \{0,1\}$ indicating whether H has a low-energy ground state.

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Physics: computing AdS/CFT map, decoding black-hole radiation

What can complexity theory say about the hardness of these **inherently quantum** problems?

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Before we continue:

1-minute detour for quantum computing 101

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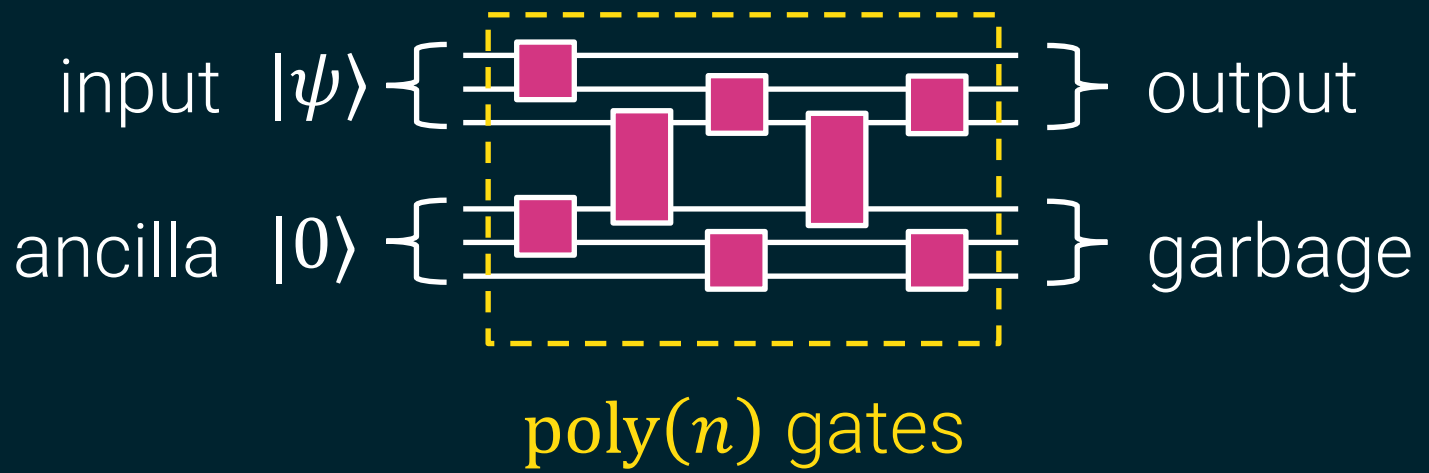

Now back to:

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The Unitary Synthesis Problem [AK06]:

Is there a reduction that works for every U?

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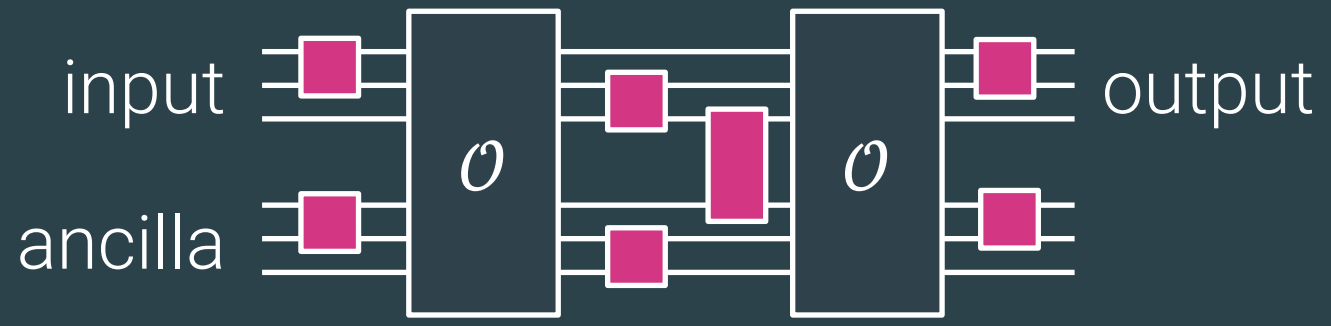

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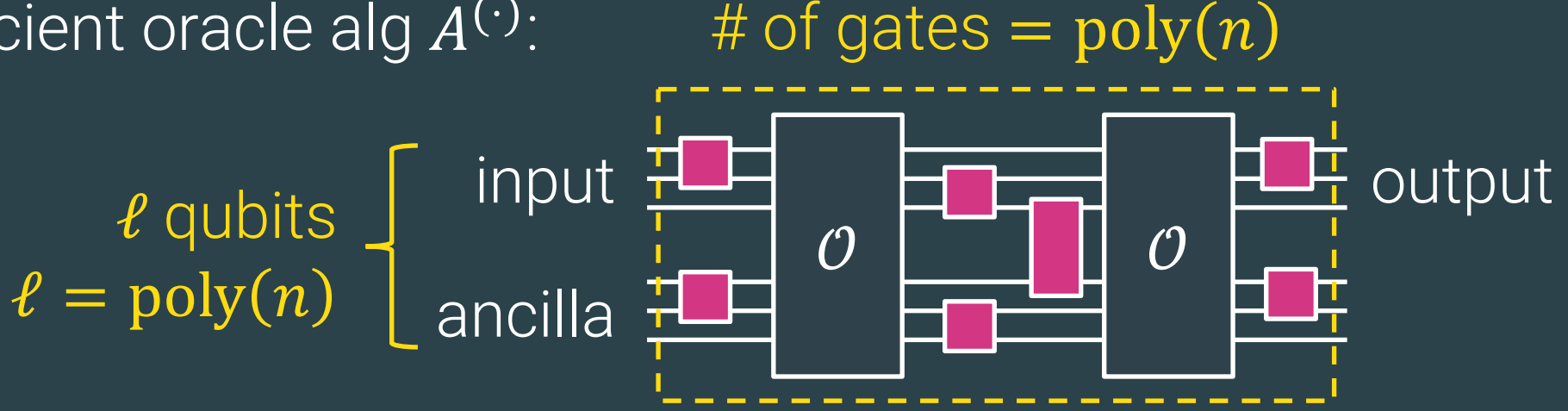

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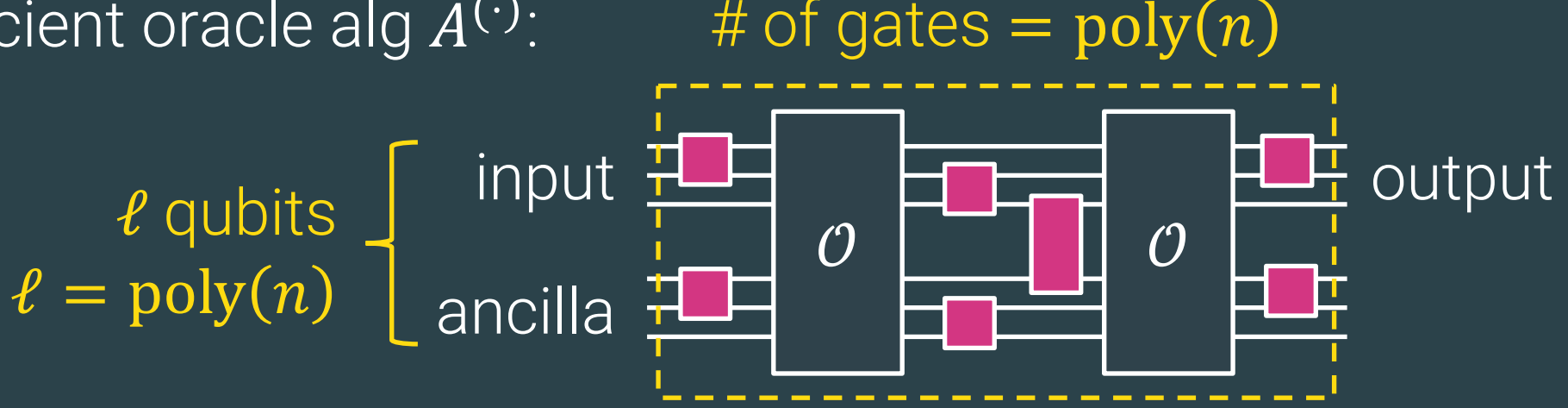

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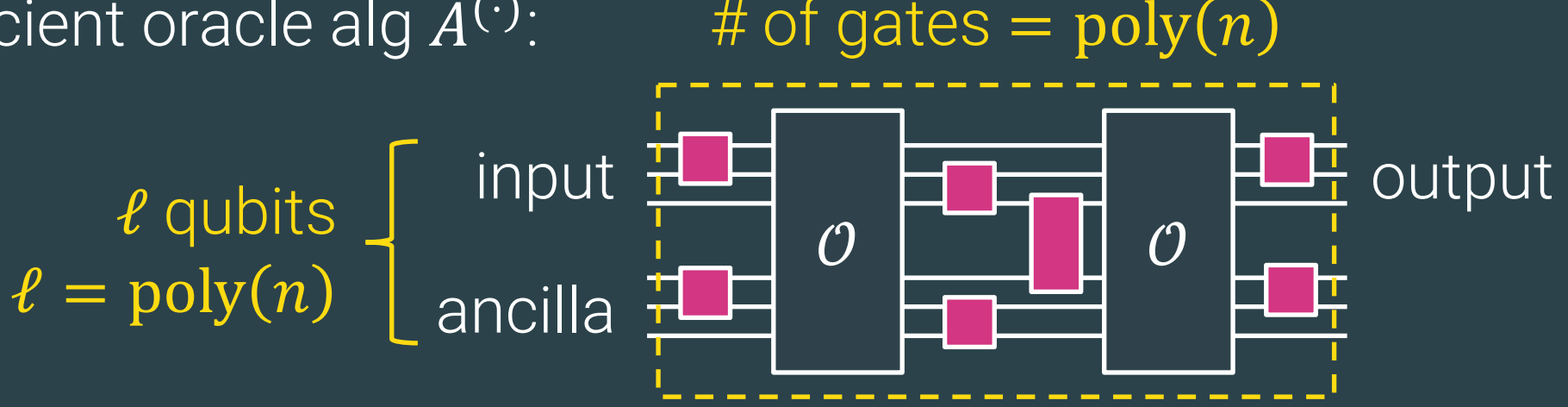

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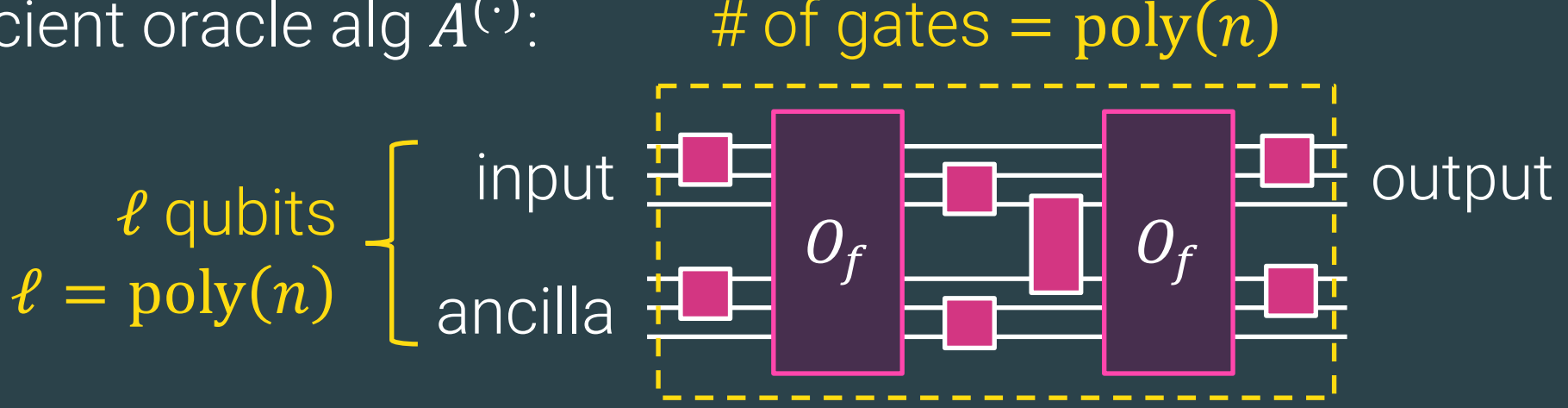

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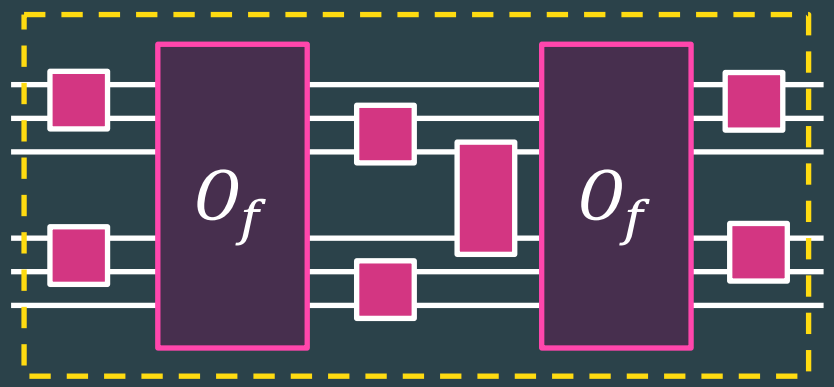

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  Note: [AK06] prove a 1-query lower bound for a very special class of oracle algorithms.

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In fact, they can solve any **classical input**, **quantum output** problem.
[Aar16, INNRY22, Ros23]

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Note: when $\ell = 2^{2n}$, possible to learn description of U in one query.

Rest of this talk

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Our answer: possibly harder than computing any function!

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Note: this result implies our unitary synthesis lower bound.

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given one query to a function f, which can depend on $R := \{R_k\}$.

Next up: what does a one-query adversary look like?

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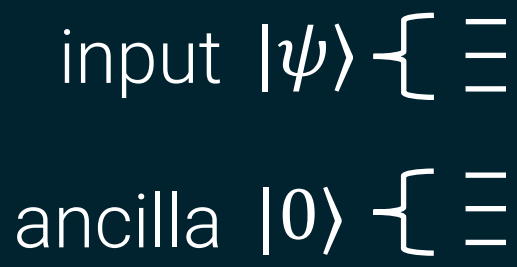

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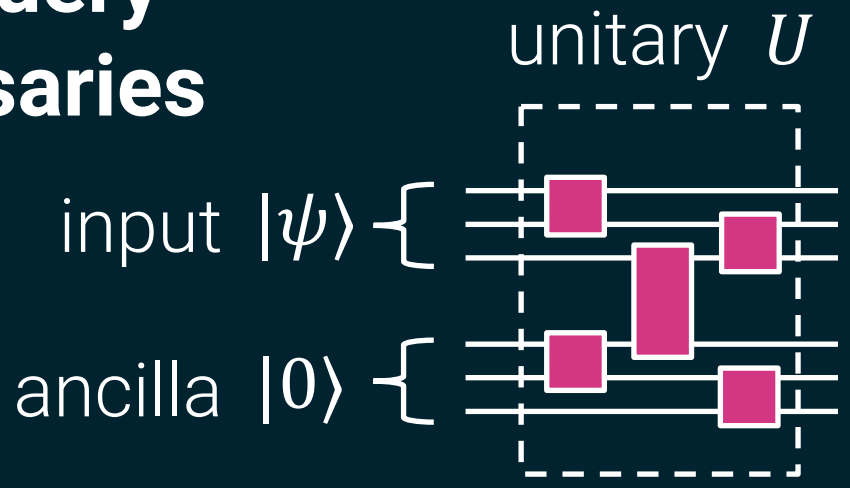

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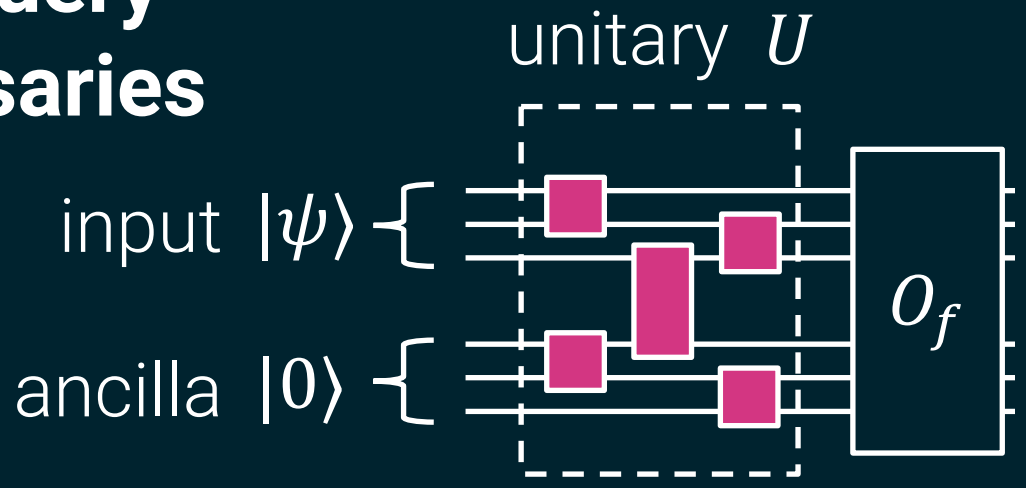

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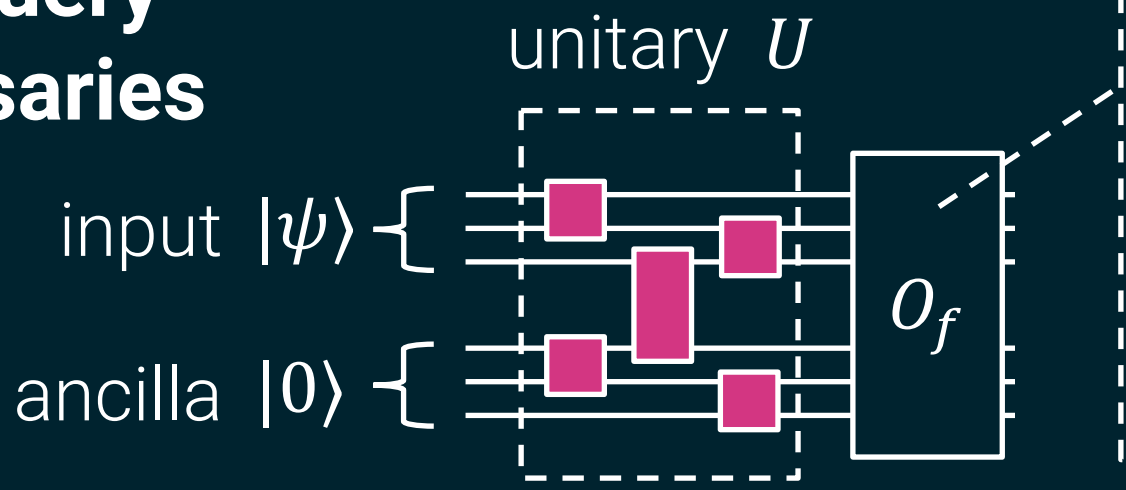

$$O_f = \begin{pmatrix} \ddots & & \\ & f(z) & \\ & & \ddots \end{pmatrix}$$

$2^\ell \times 2^\ell$ diagonal matrix, z-th entry is $f(z) \in \{\pm 1\}$

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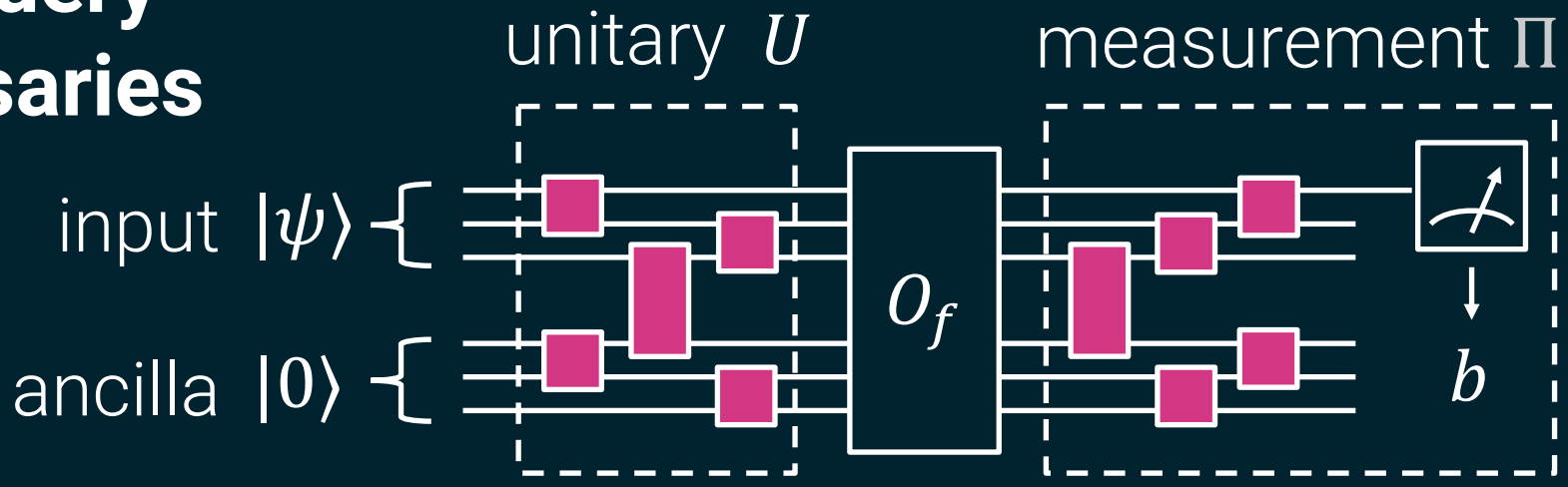

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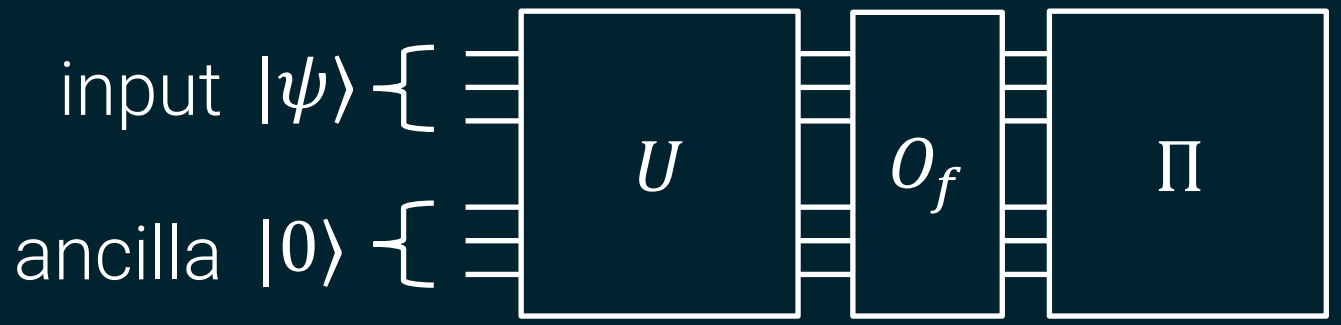

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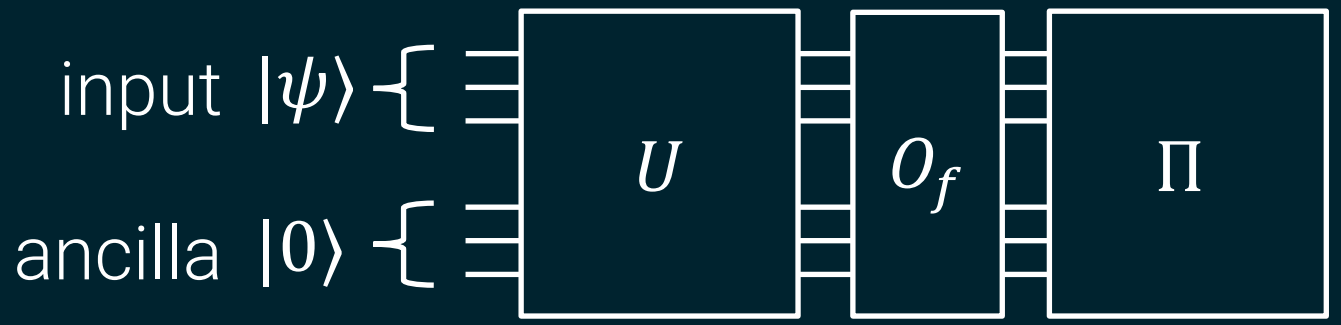

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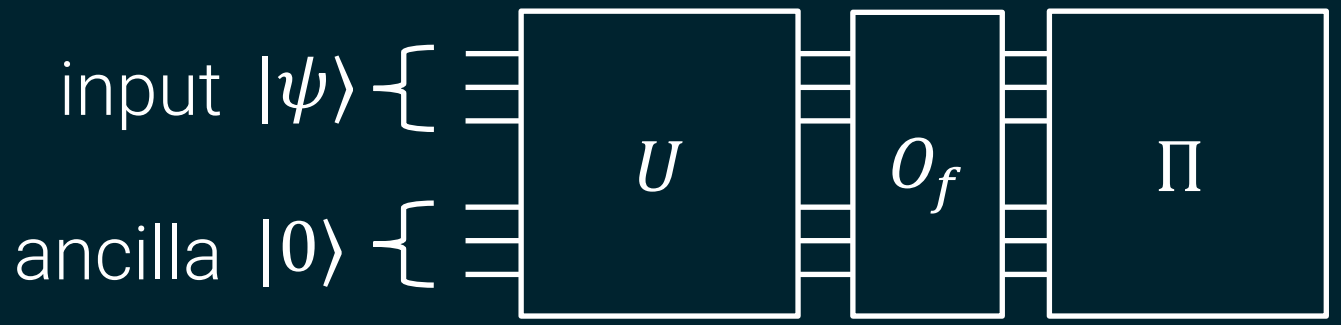

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Adversary's **distinguishing advantage** for fixed R is

$$\mathop{\mathbb{E}}_{k \leftarrow [K]} \Pr[A^f(|\psi_{R_k}\rangle) \text{ outputs } 1] - \mathop{\mathbb{E}}_{h} \Pr[A^f(|\psi_h\rangle) \text{ outputs } 1]$$

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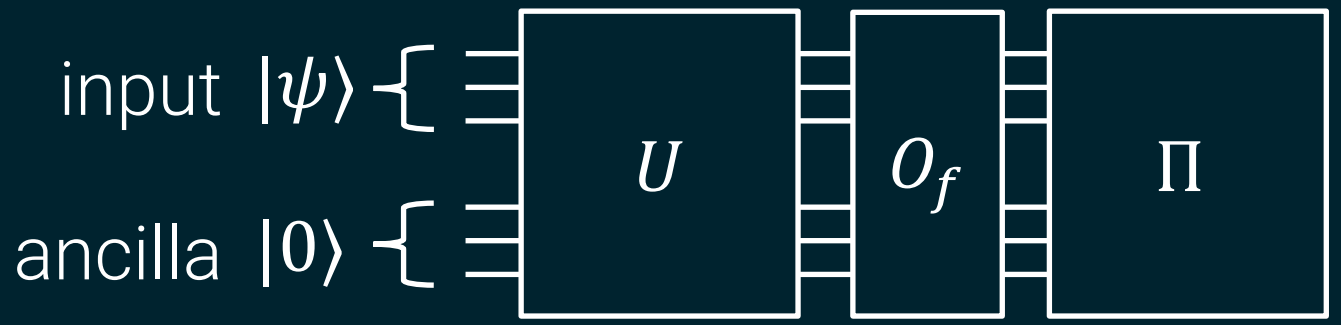

$$\Pr[A^f(|\psi\rangle) \text{ outputs } 1] = \left\| \Pi \cdot O_f \cdot U \cdot |\psi\rangle|0\rangle \right\|^2$$

Adversary's **distinguishing advantage** for fixed R is

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Rest of this talk

Part 1:
Connect unitary synthesis to breaking quantum cryptography

Part 2:
A special case of our proof

A special class of one-query adversaries

Assume adversary sets $\ell = n$ (no ancillas) and $U = \mathrm{Id}$.

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Disclaimer: We can rule out these attacks with a counting argument, but today we'll see a different proof.

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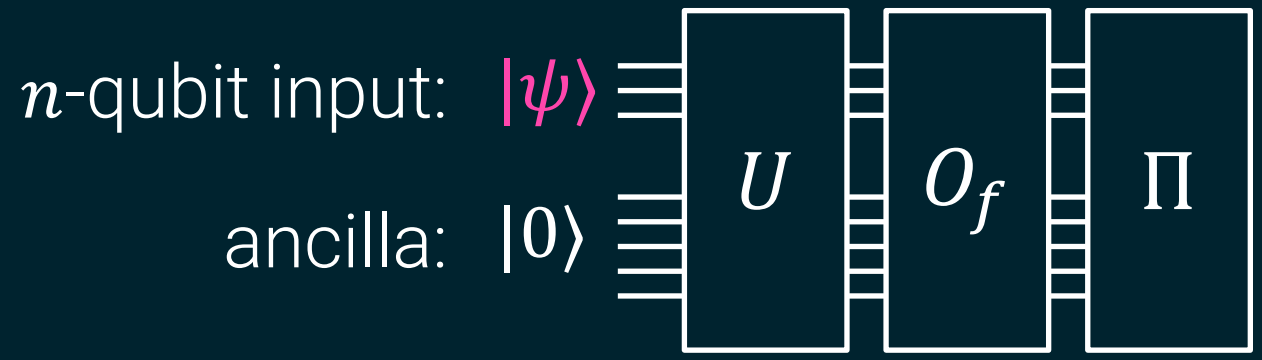

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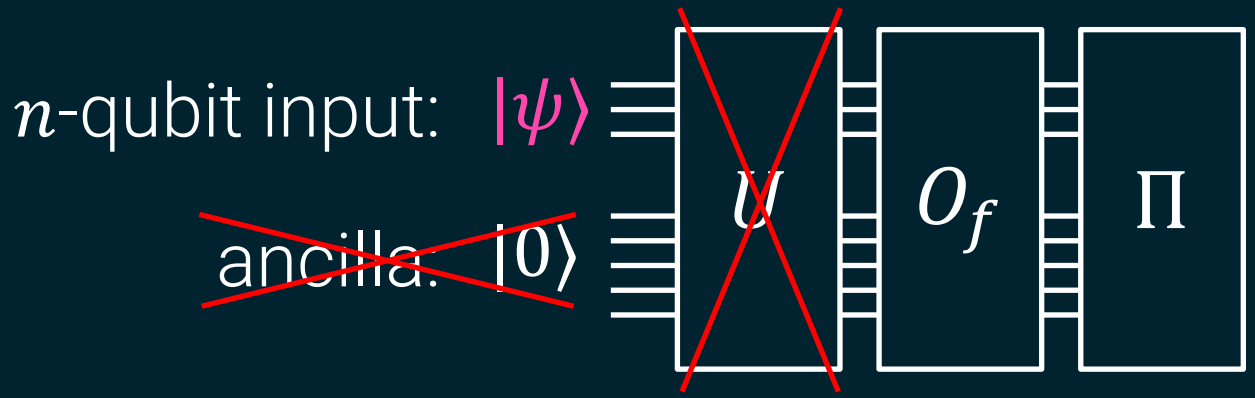

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Adversary's advantage (for this special class):

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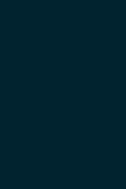

?

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Key step: we can refactor this as $\langle v_f | \cdot (\text{random matrix}) \cdot |v_f\rangle$

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Then matrix Chernoff will bound the max over all unit vectors.

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Since all the terms look identical, it suffices to just look at one term.

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$N \times N$ diagonal matrix, x-th entry is $R_k(x)$

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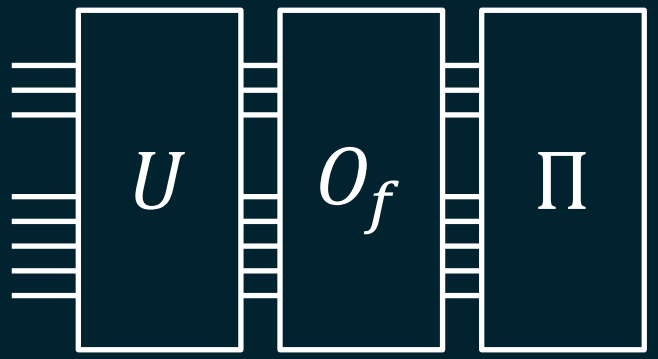
by Matrix Chernoff with $X_k = D_{R_k}\cdot\Pi\cdot D_{R_k}$

How do we handle general one-query adversaries?

General one-query adversaries

n qubit input: $|\psi_h\rangle$

ancilla: $|0\rangle$



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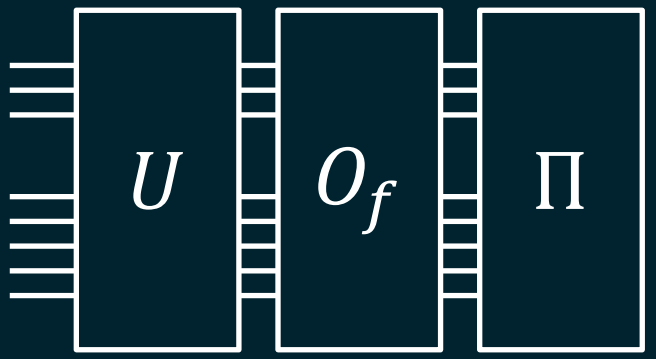

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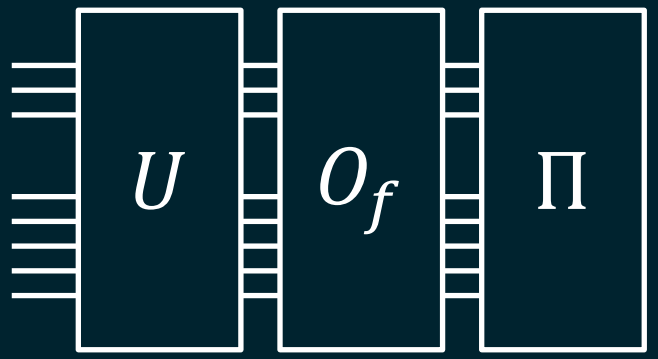

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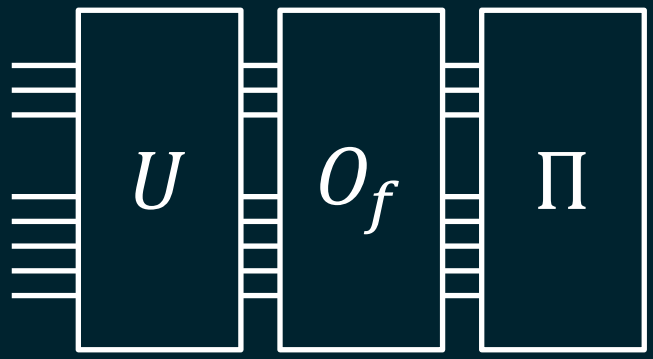

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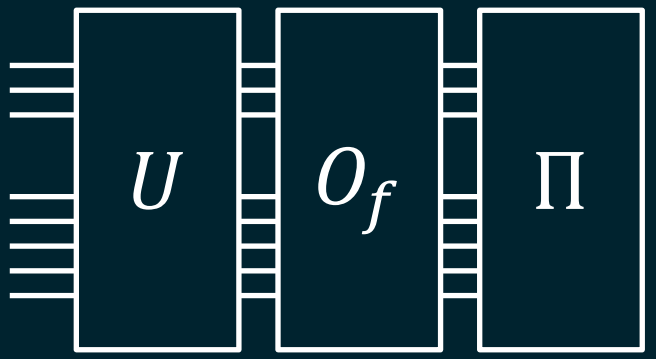

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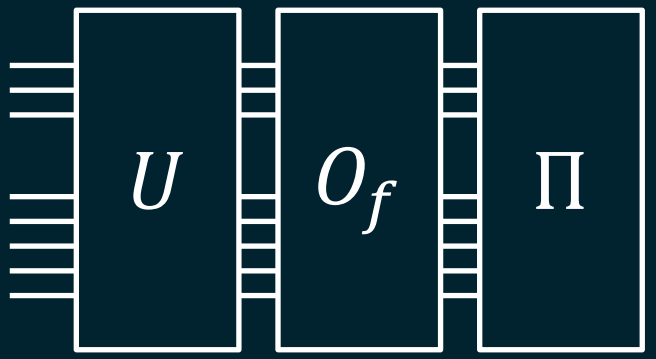

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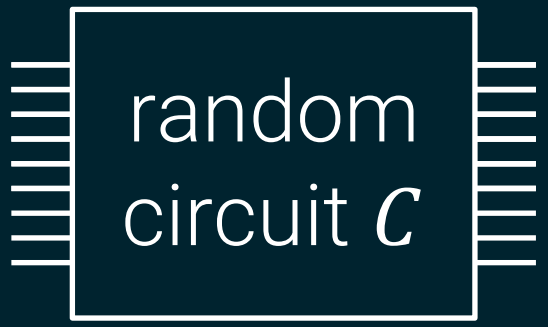

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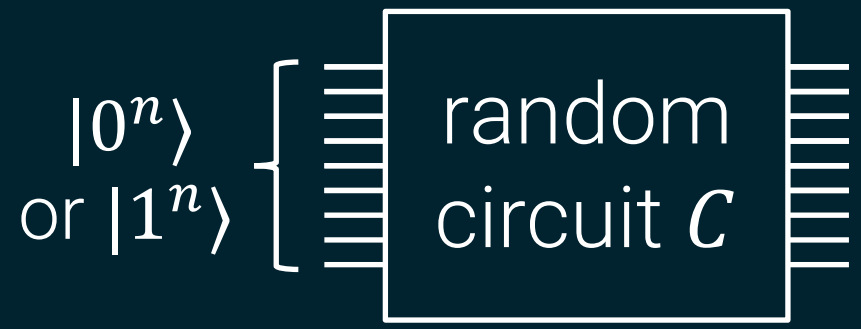

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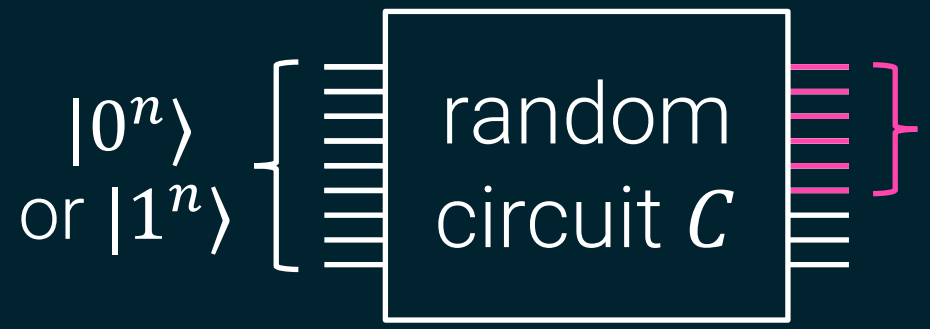

Task: given description of C and $2n/3$ qubits of $C|b^n\rangle$, determine b.

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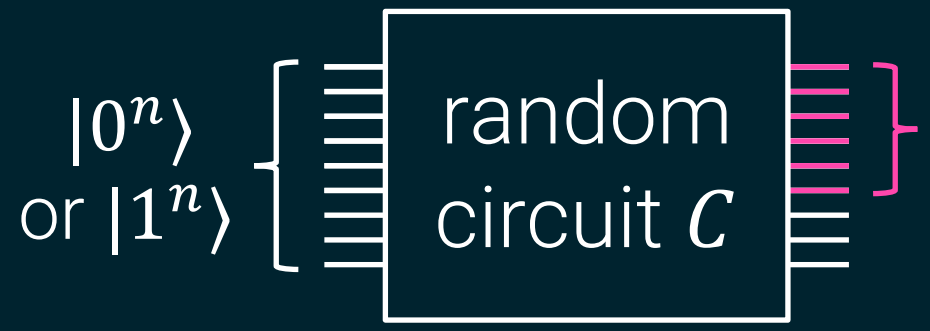

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Is this easy given a halting oracle?

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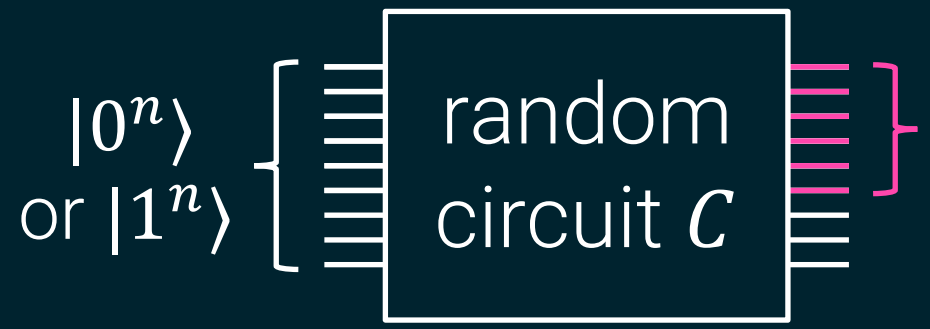
Open problem #1: prove that our PRS distinguishing game is hard even given $\mathbf{poly}(n)$ queries to an arbitrary f.

Open problem #2:



Task: given description of C and $2n/3$ qubits of $C|b^n\rangle$, determine b.

Is this easy given a halting oracle?

Thanks for listening!