How to Construct Random Unitaries

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Berkeley → NYU

joint work with Hsin-Yuan Huang

Haar measure: uniform distribution on unitaries

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Property: for any unitary W, if $U \sim \text{Haar}$, $W \cdot U \sim \text{Haar}$

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Haar-random unitaries show up everywhere:

- black hole information scrambling
- entanglement
- quantum learning algorithms
- quantum crypto
- random quantum circuits
- unitary complexity
- quantum error correction

…

Challenge:

Haar-random unitaries are exponentially complex

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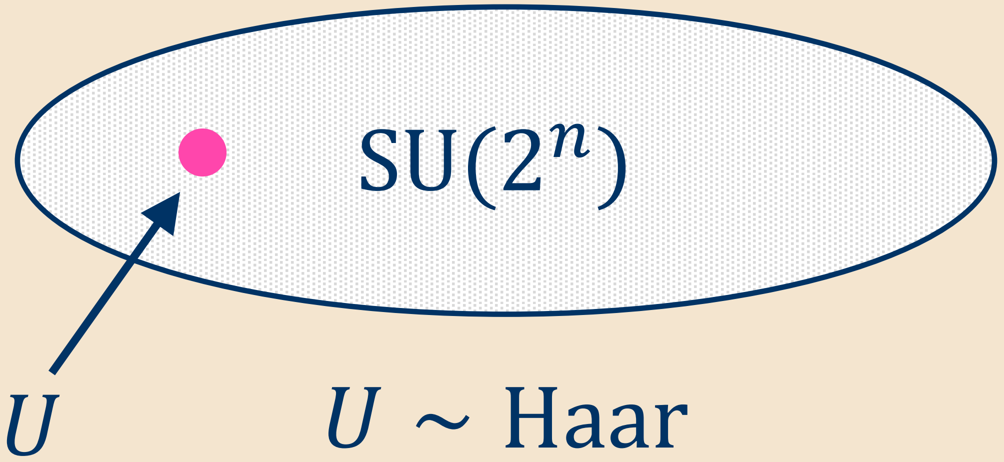

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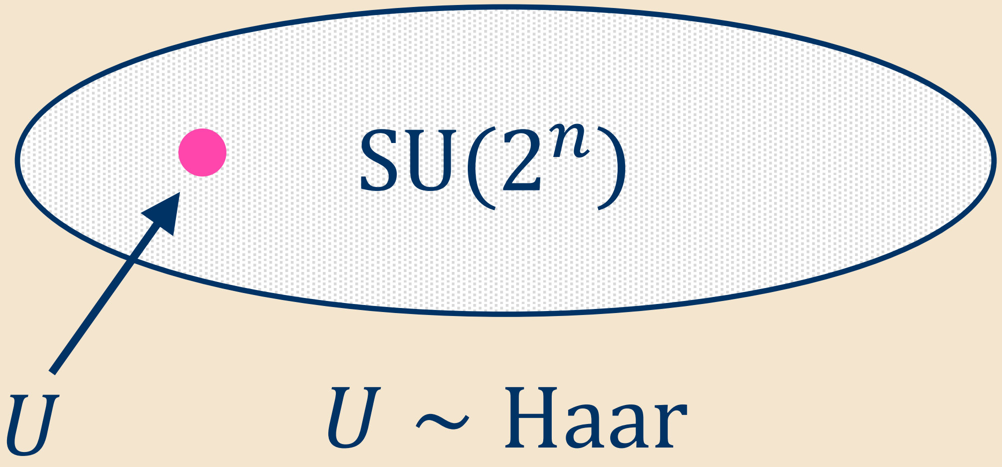
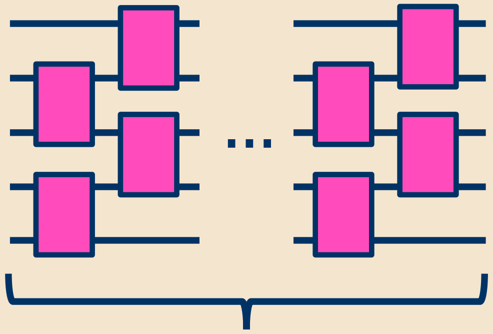


$U \sim \text{Haar}$

minimal circuit for U



$\exp(n)$

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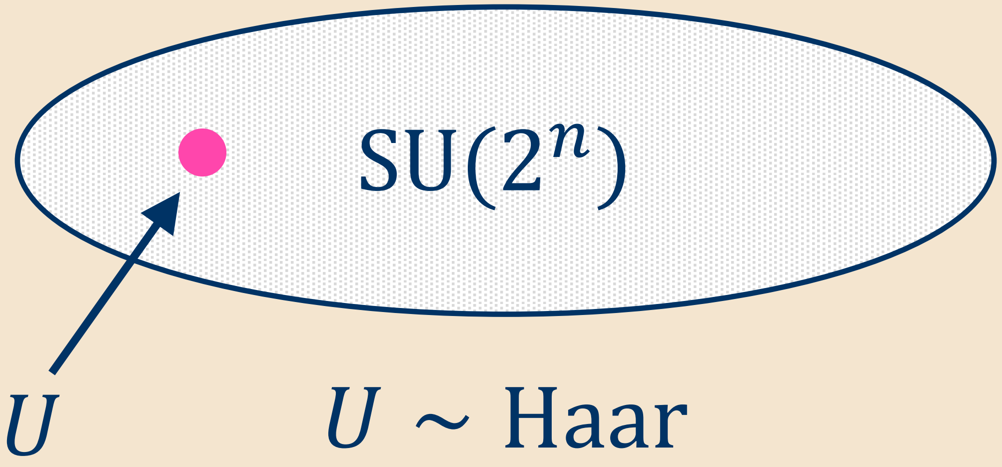

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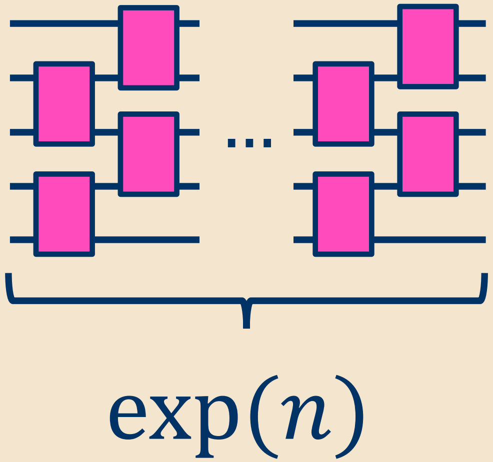

This makes them impractical for most applications!

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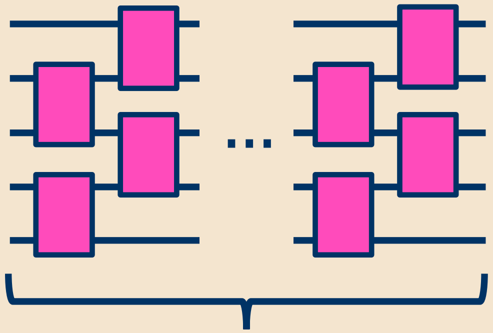

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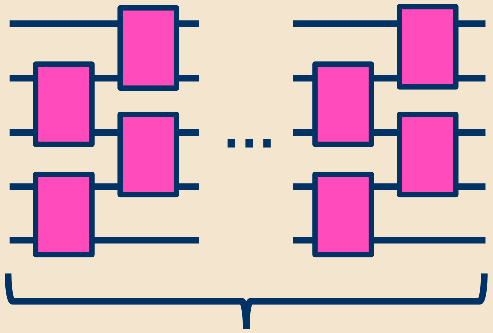

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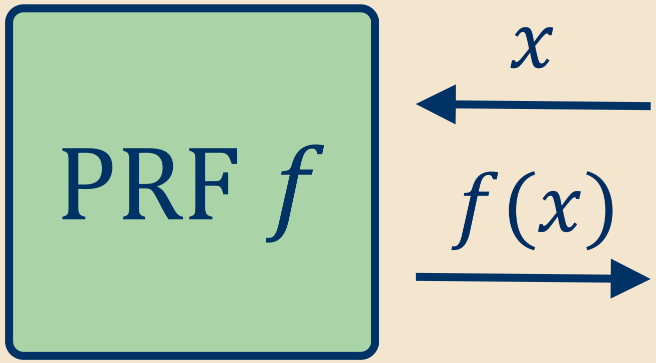

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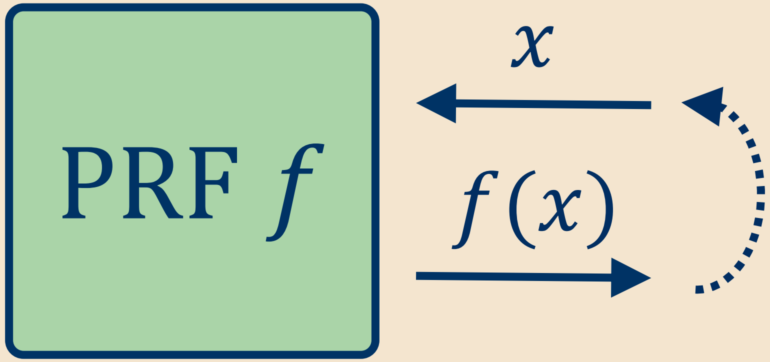

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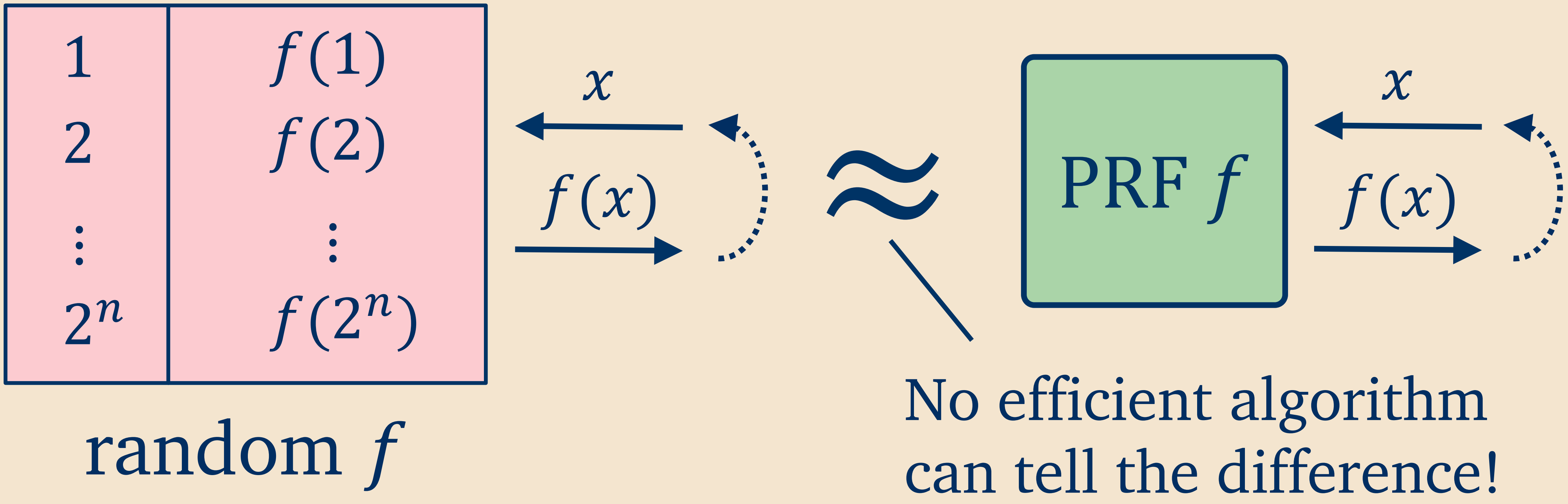

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efficiently-computable unitaries that appear Haar-random

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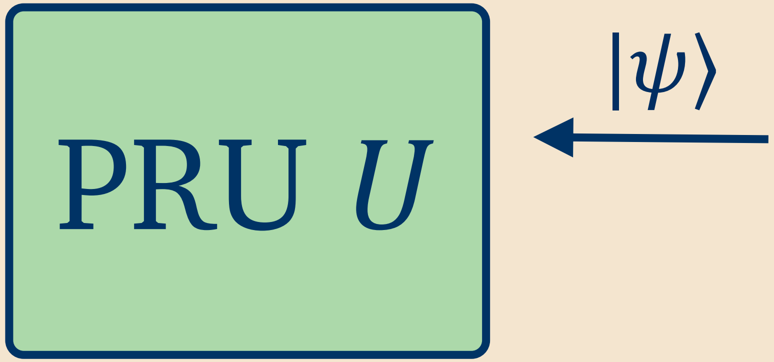

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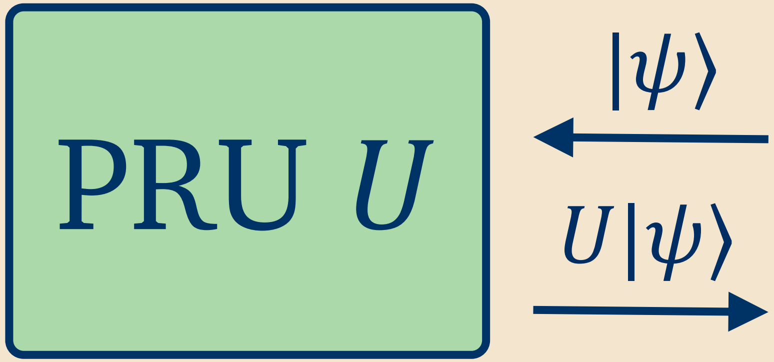

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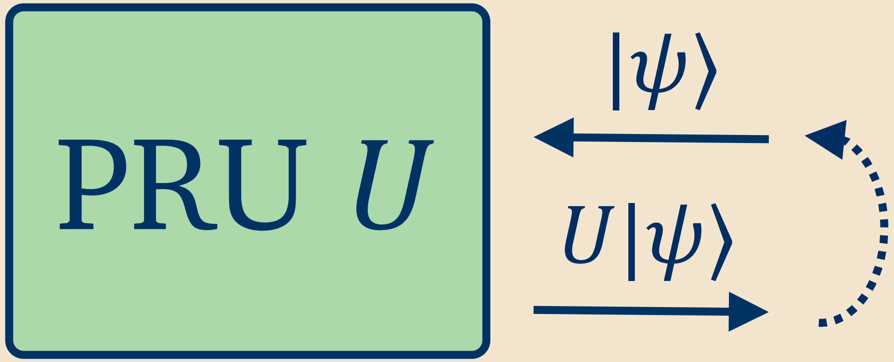

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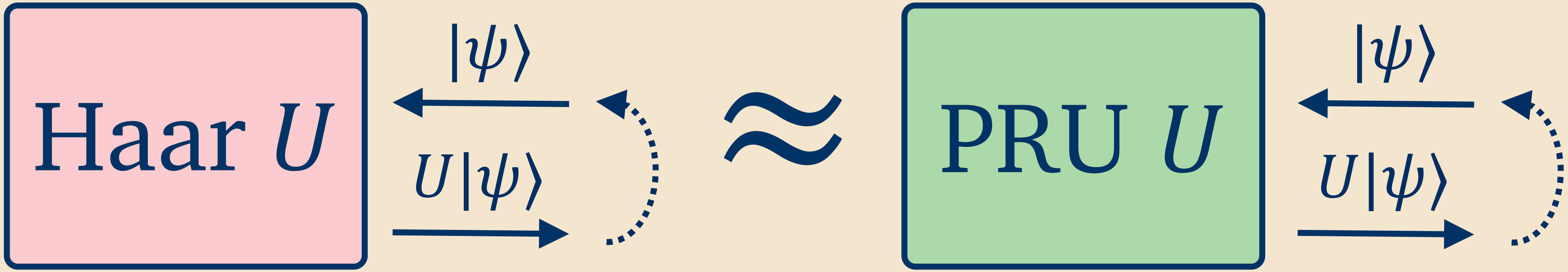

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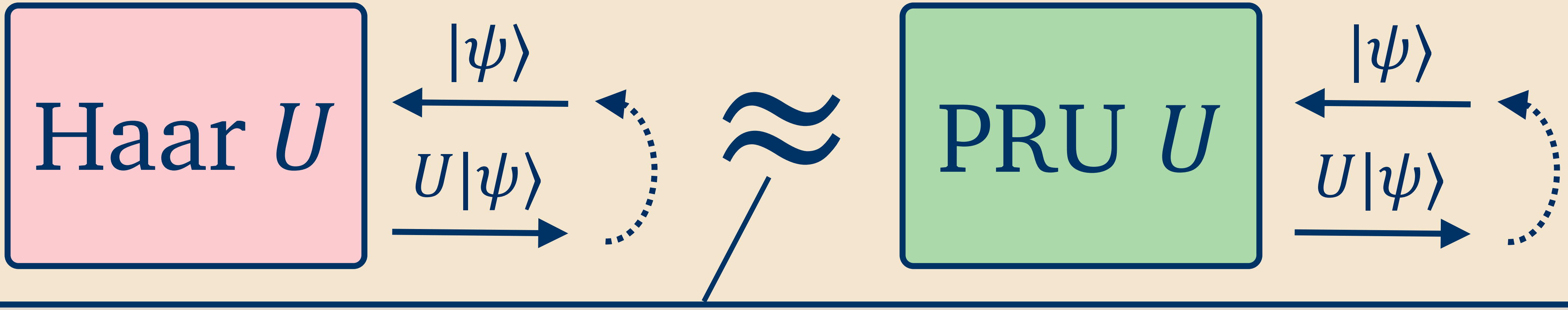

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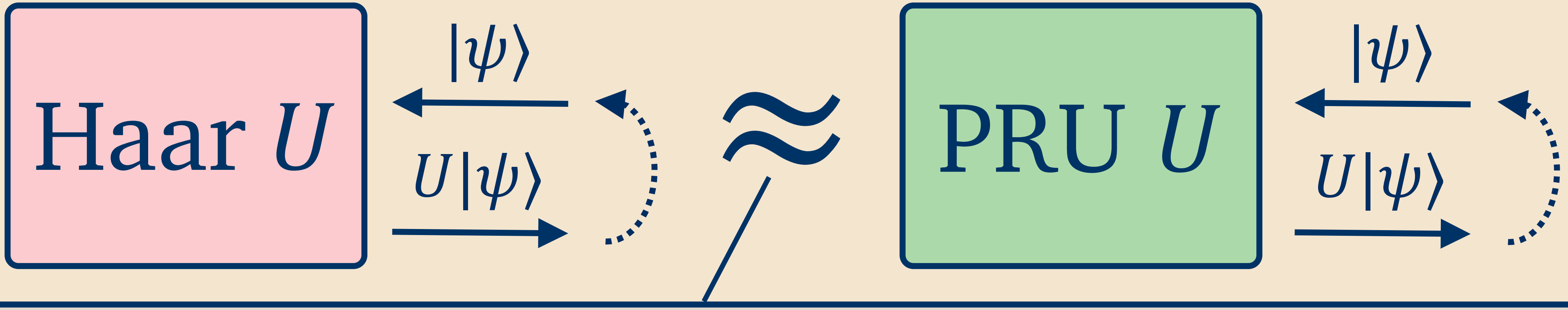

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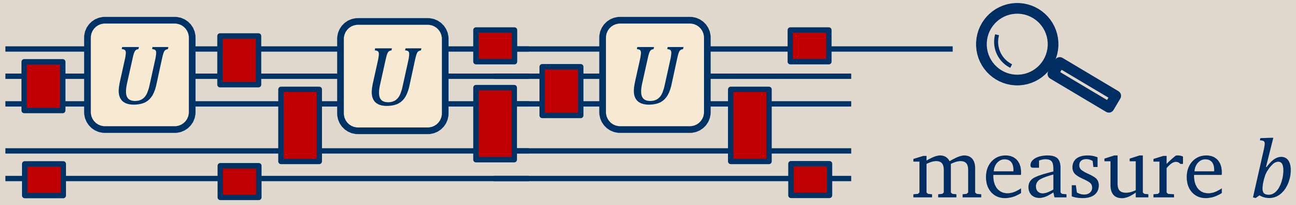

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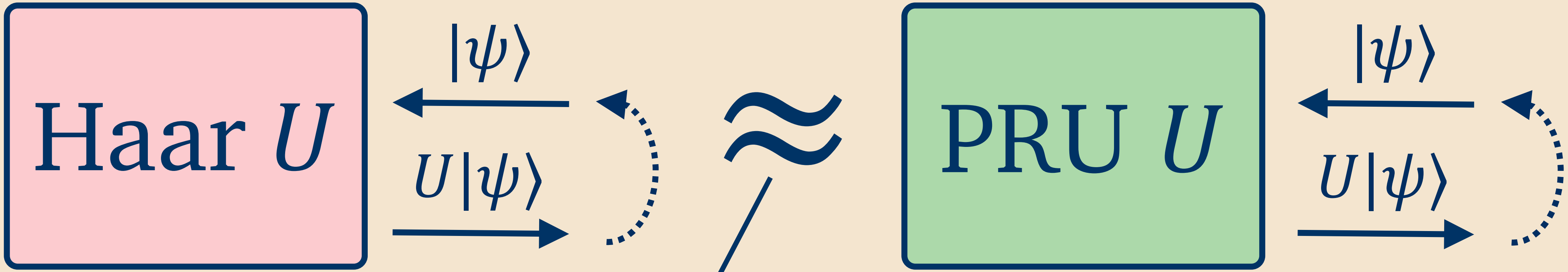

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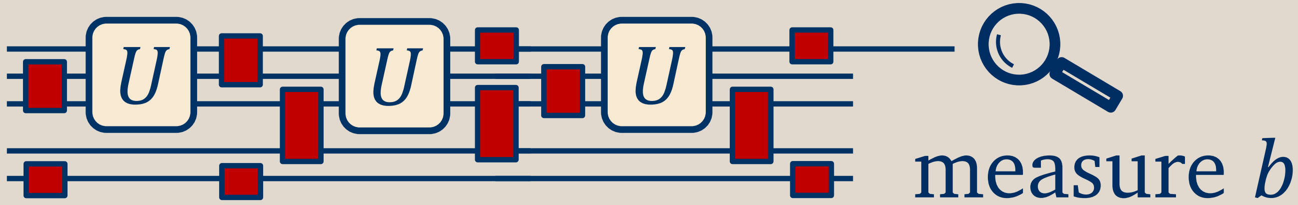

$$\Pr[b = 1 \mid U \leftarrow \text{Haar}] \approx \Pr[b = 1 \mid U \leftarrow \text{PRU}]$$

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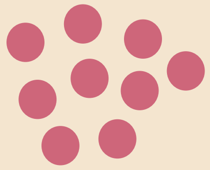

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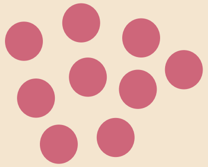

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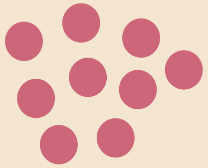

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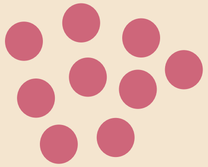

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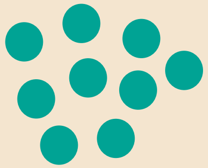

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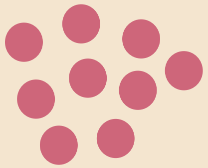

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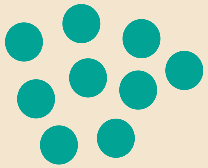

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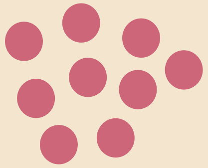

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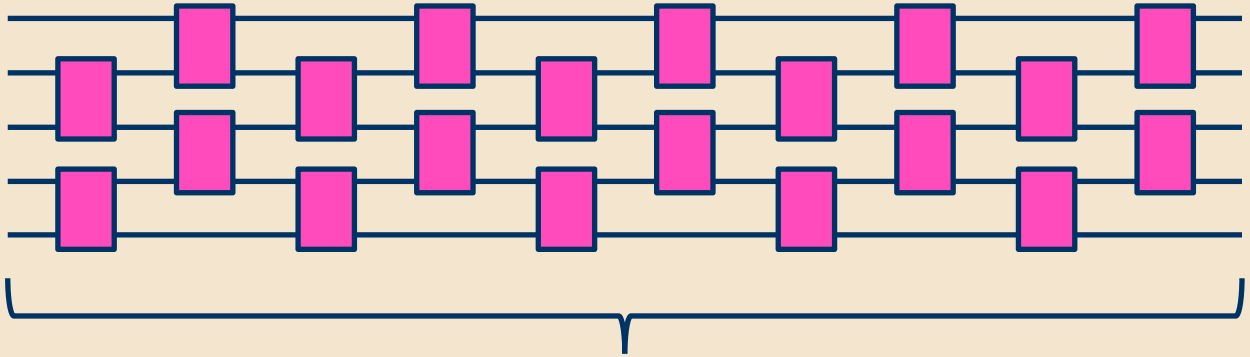
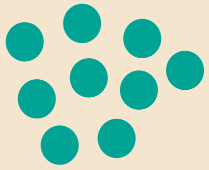


size = poly(# of qubits)



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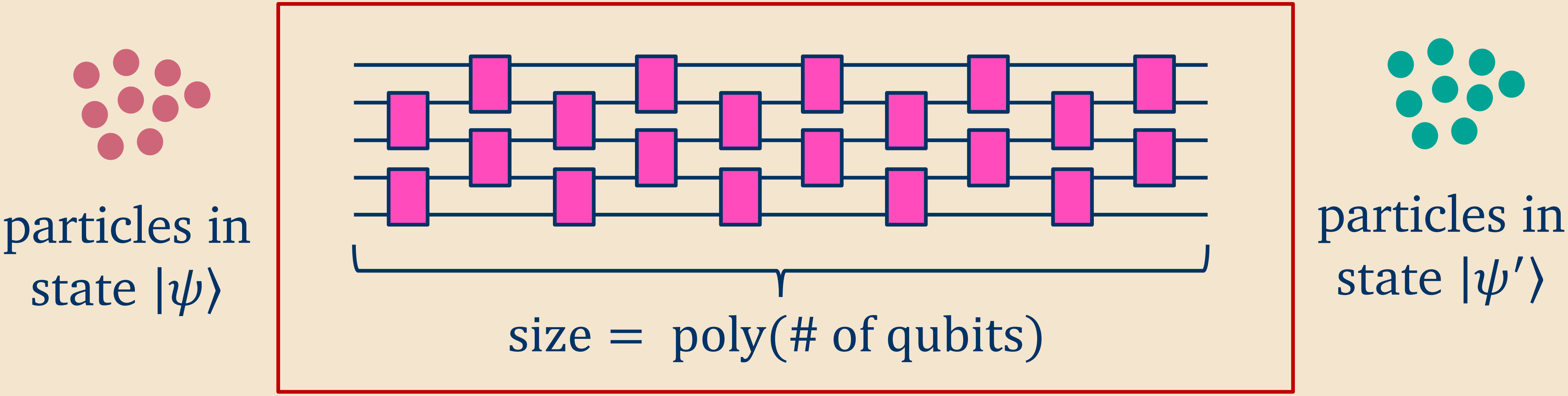

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Consequence: many physics results now rely on the assumption that various physical processes are PRUs [KP23,YE23,EFLVY24]

But do PRUs exist?

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This was left as an open problem by [JLS18].

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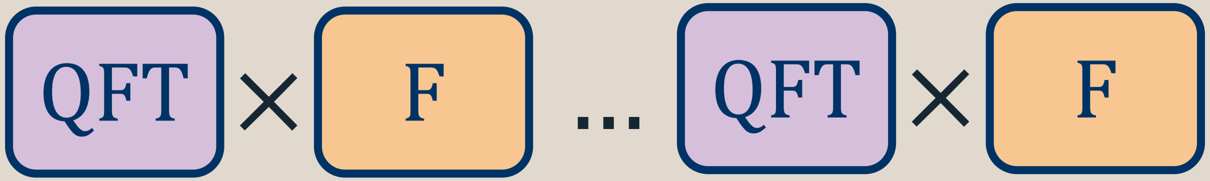

[JLS18]

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P × F × C (C = Clifford)

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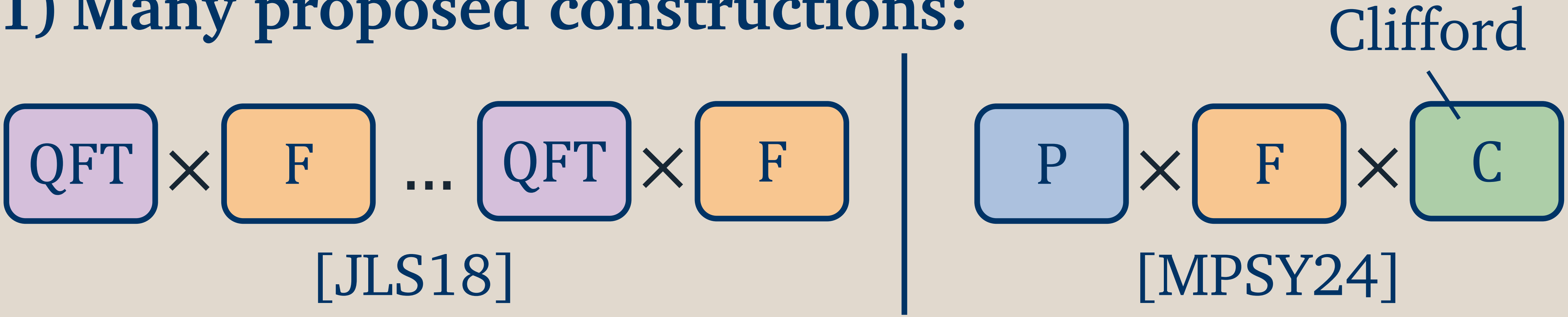

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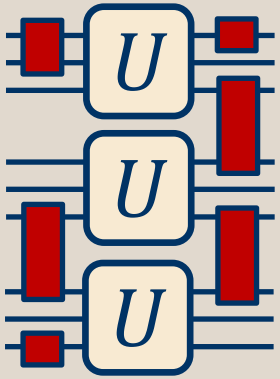

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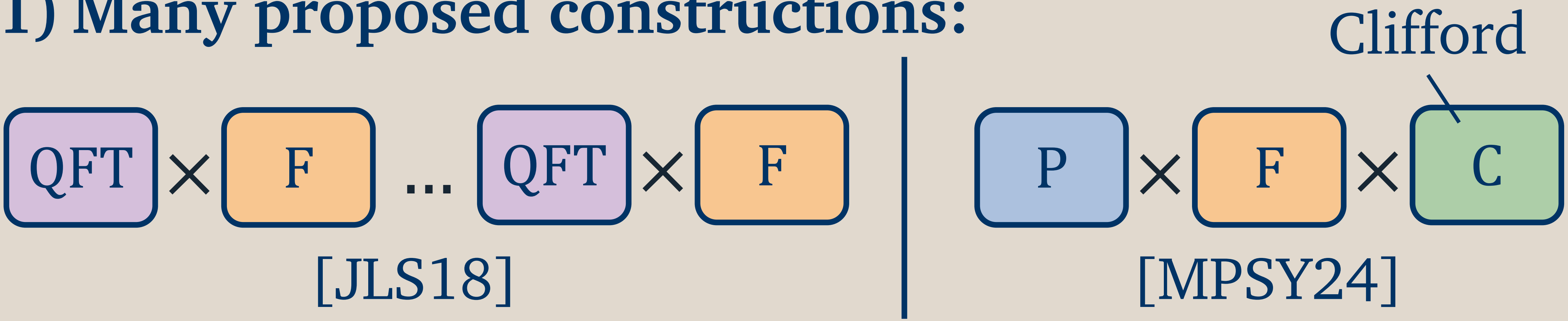

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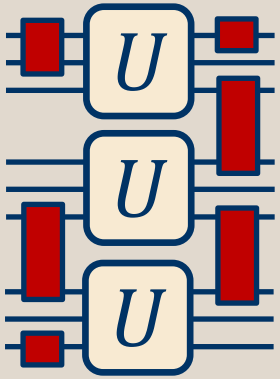

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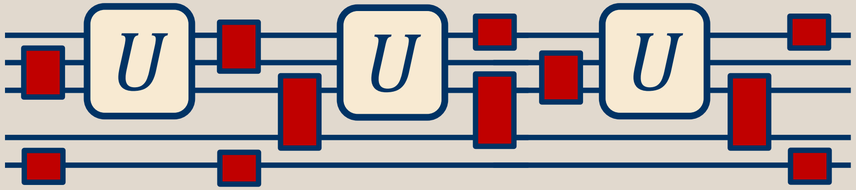

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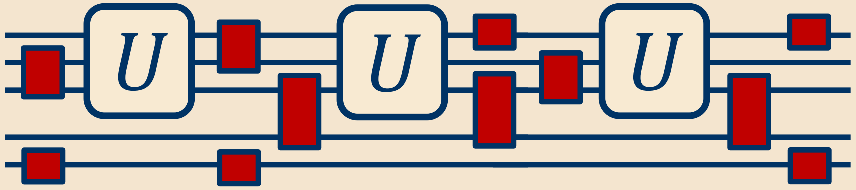

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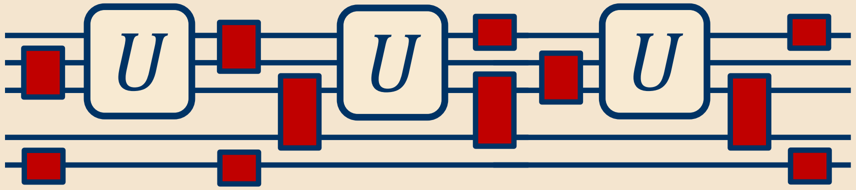

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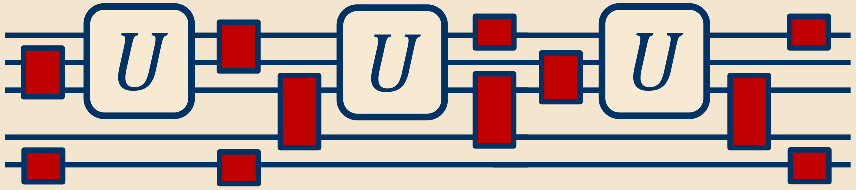

2) Mathematics of random unitaries is complicated.

- Weingarten calculus
- free probability
- ???

Theorem 3.1. *Let* k *be a positive integer. For any permutation* $\sigma \in \mathcal{S}_k$ *and nonnegative integer* g*, we have*

$$(k-1)^g \#P(\sigma, |\sigma|) \leq \#P(\sigma, |\sigma| + 2g) \leq (6k^{7/2})^g \#P(\sigma, |\sigma|).$$

Theorem 3.2. *For any* $\sigma \in \mathcal{S}_k$ *and* $d > \sqrt{6}k^{7/4}$,

$$\frac{1}{1 - \frac{k-1}{d^2}} \leq \frac{(-1)^{|\sigma|} d^{k+|\sigma|} \mathrm{Wg}^U(\sigma, d)}{\#P(\sigma, |\sigma|)} \leq \frac{1}{1 - \frac{6k^{7/2}}{d^2}}.$$

In addition, the l.h.s inequality is valid for any $d \geq k$.

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New technique: the path-recording oracle

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- only uses basic quantum info (purification)

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But for this talk, I'll focus on the weakest notion.

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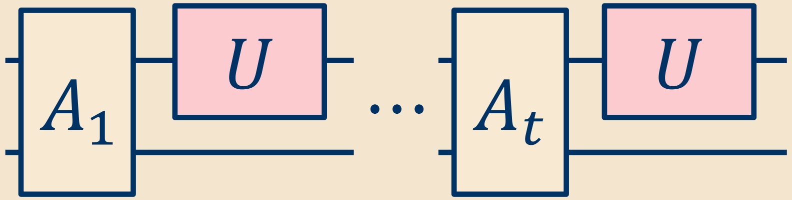

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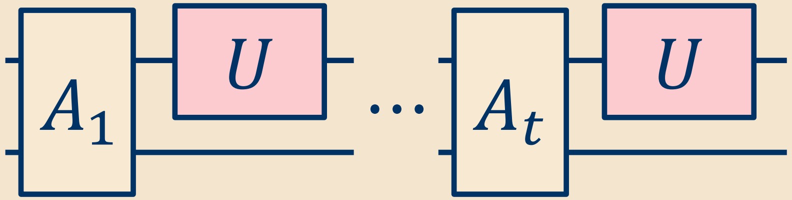

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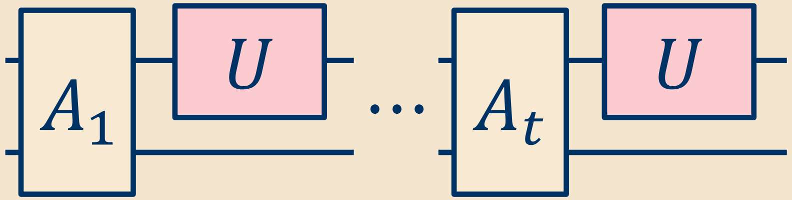

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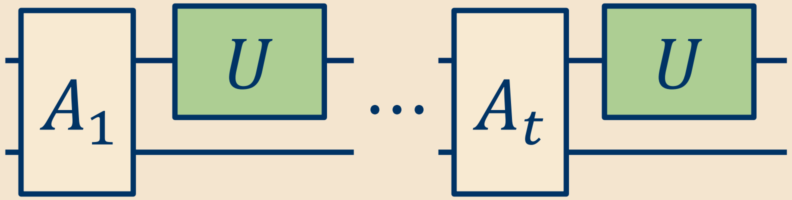

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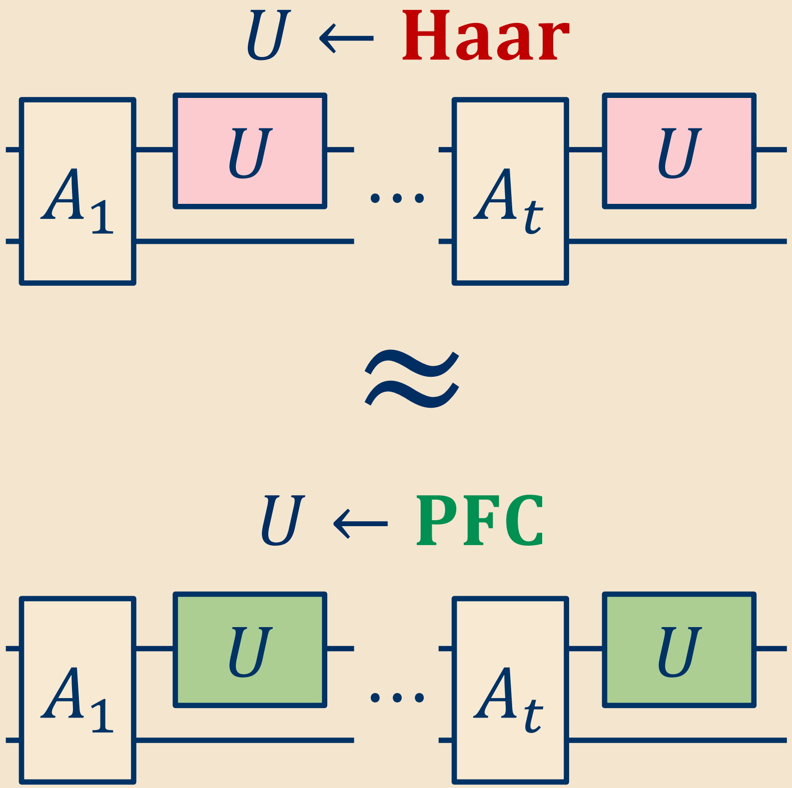

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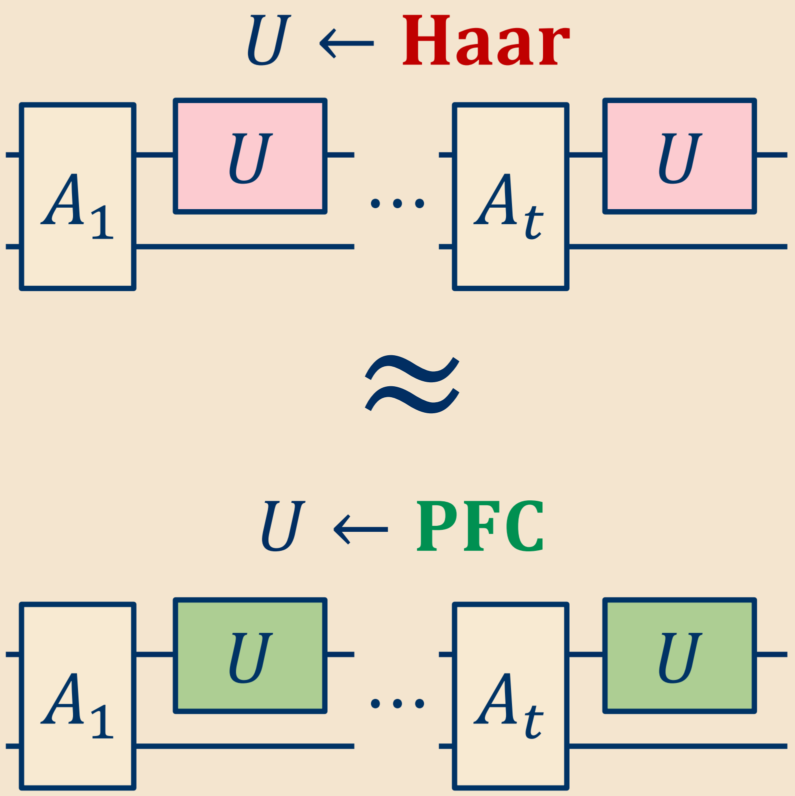

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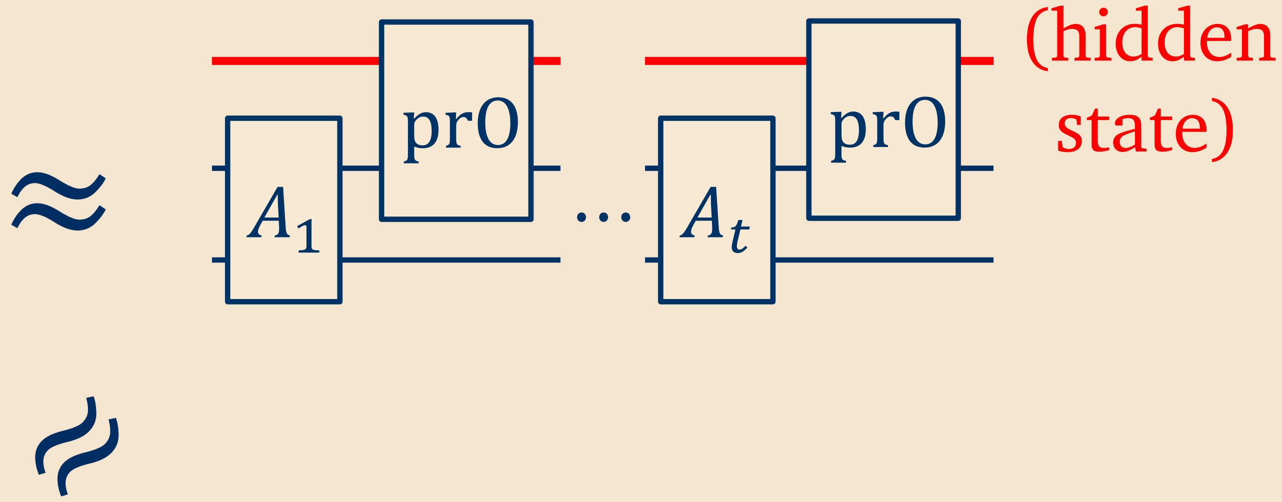

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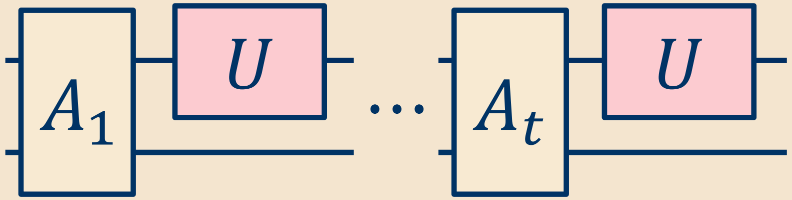

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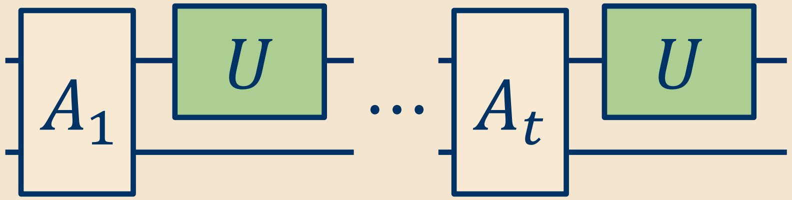

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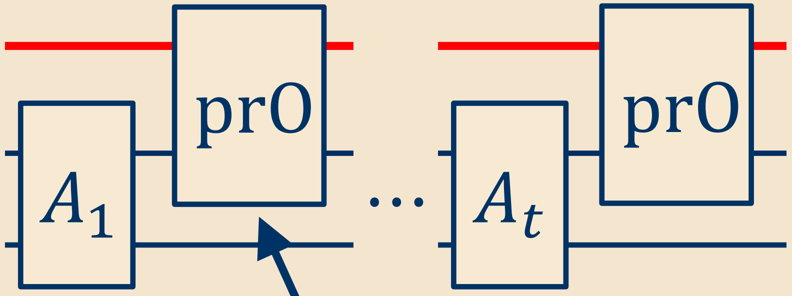

(hidden state)

$\approx$

(path-recording oracle) maintains a data structure that "lazily samples" a Haar-random unitary

Cartoon overview of our proof

Want to show:
For all efficient algorithms A,

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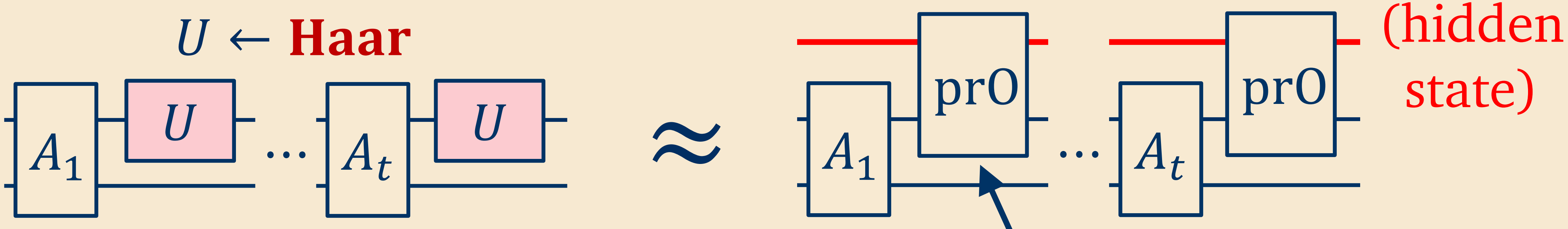
most of the proof



(path-recording oracle)
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Rest of this talk

- **Lazy sampling of a random function**
- Lazy sampling of a random unitary
- Proving correctness + PRUs exist
- Applications

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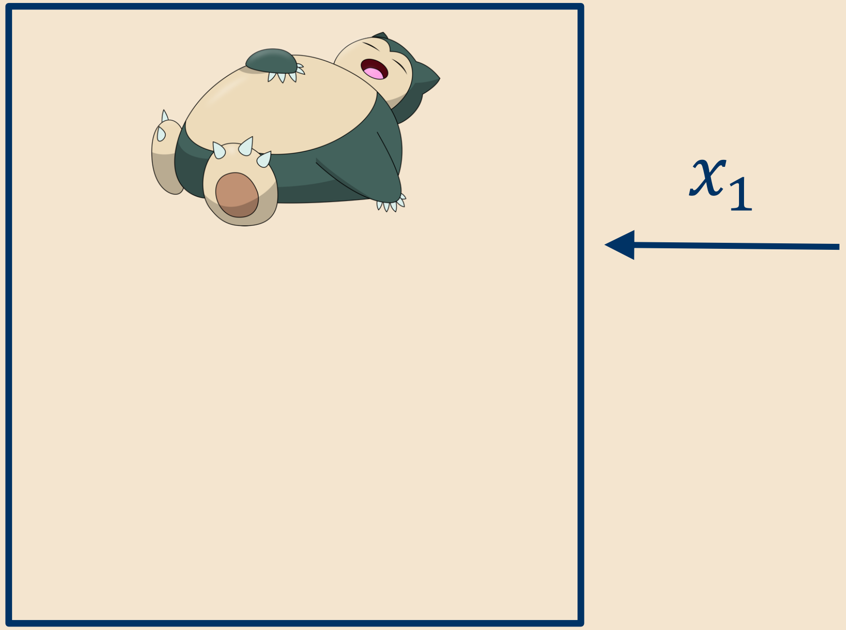

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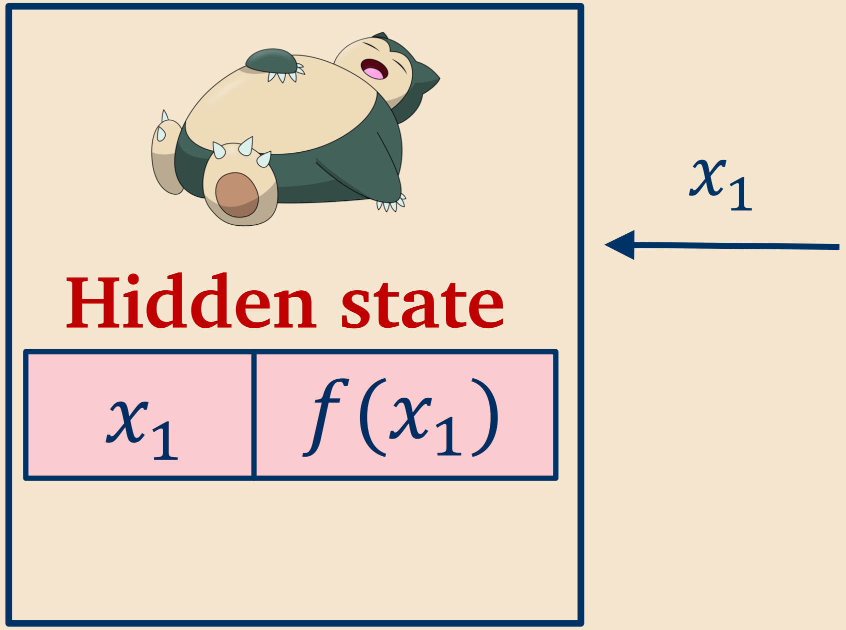

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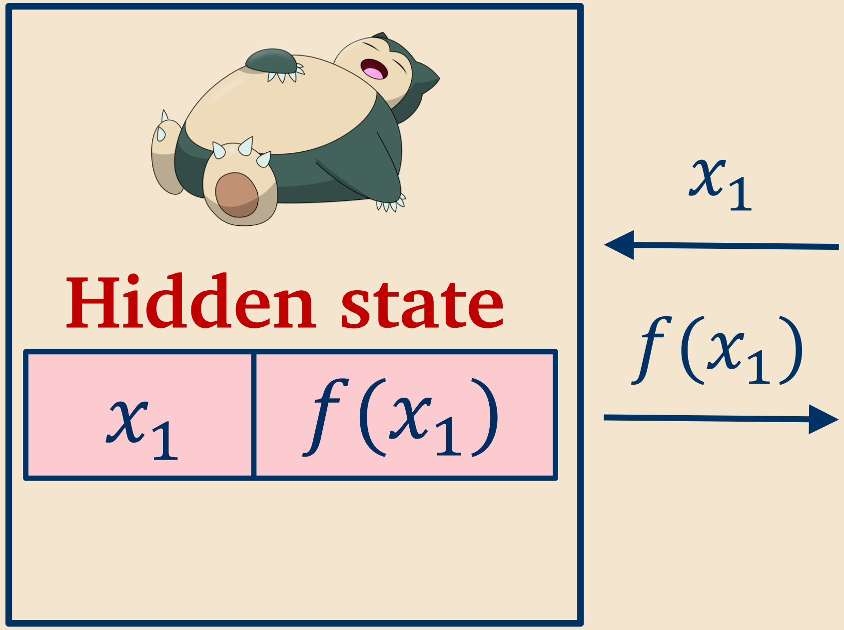

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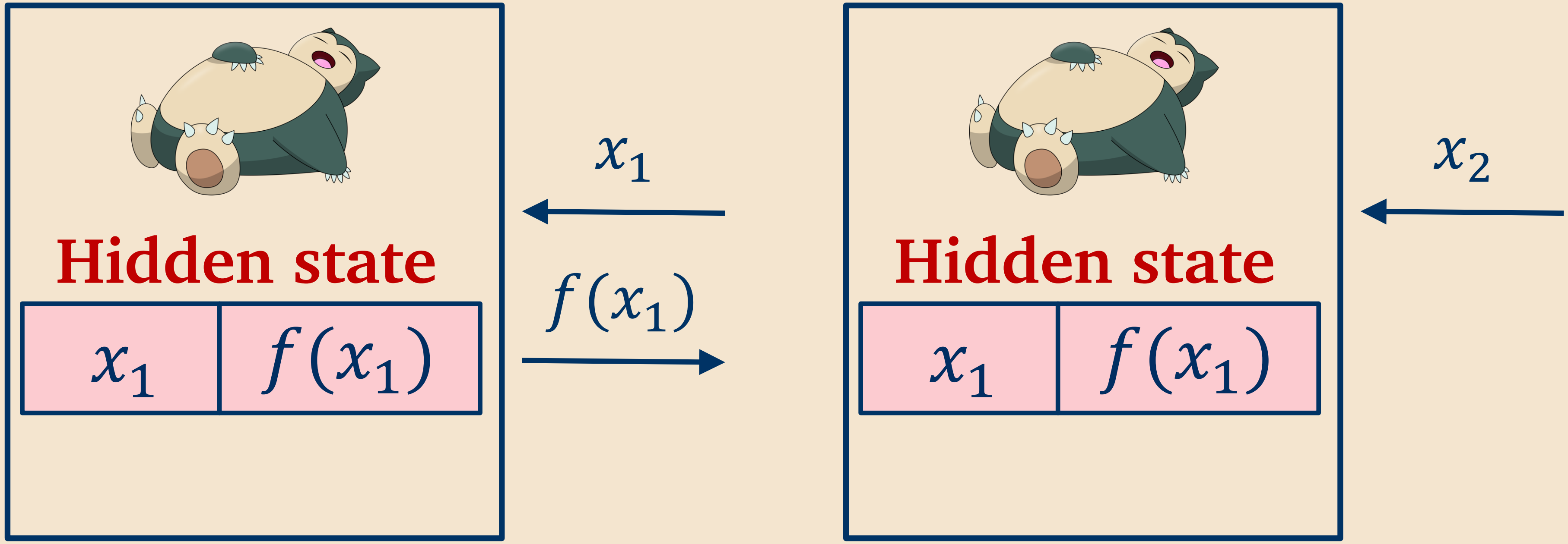

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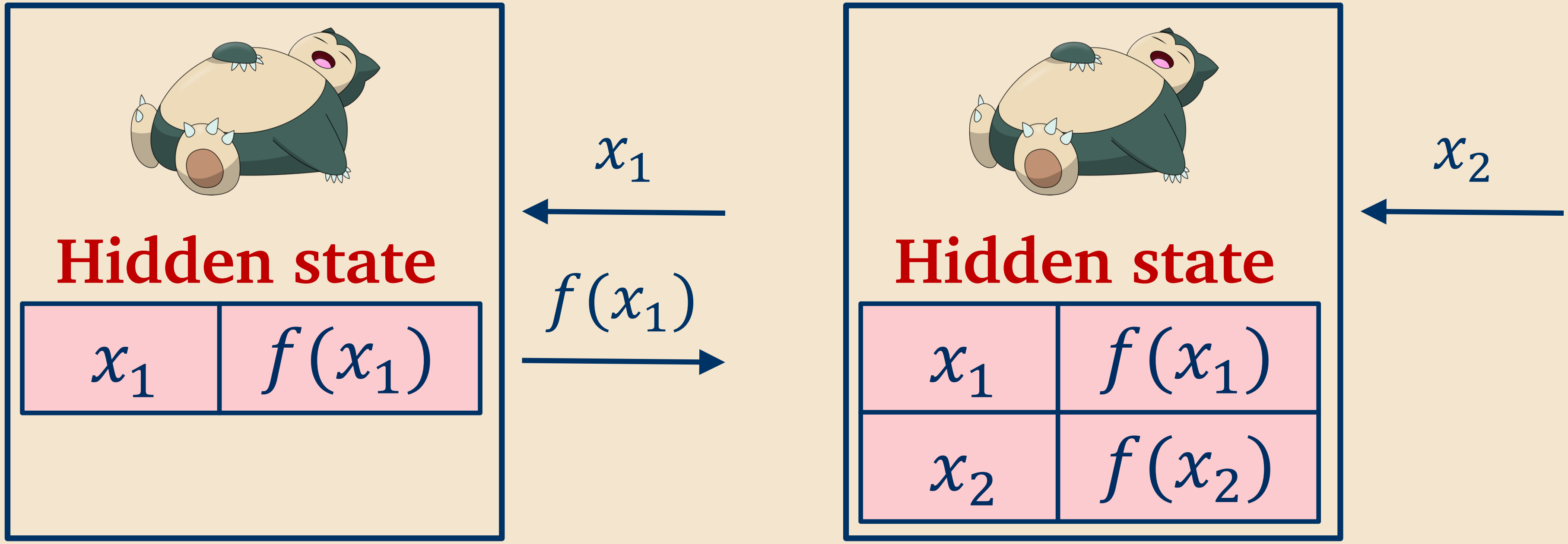

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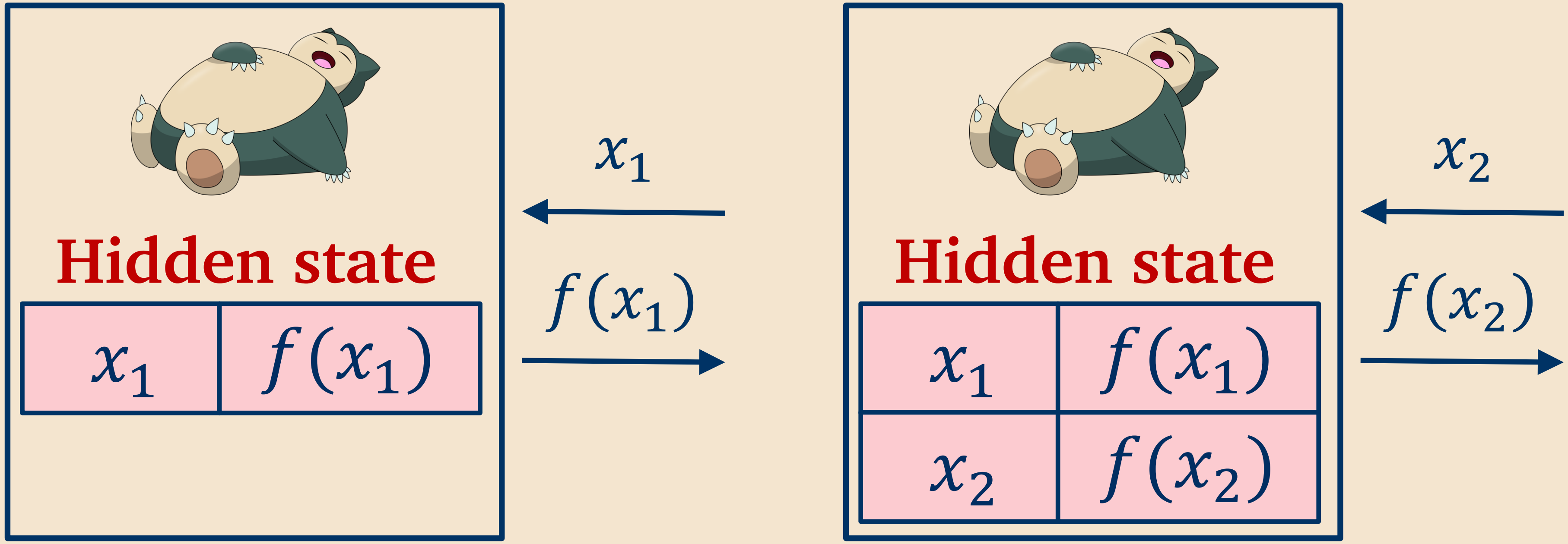

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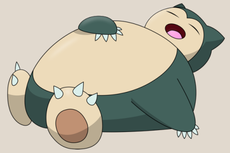

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We use **entanglement** with a **hidden data structure** that succinctly "remembers" enough information to spoof a Haar-random U.

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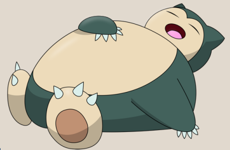

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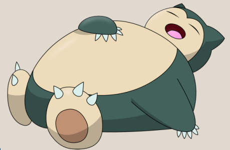

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Inspiration: compressed oracle technique [Zhandry19]

Up next:

"Derive" the path-recording oracle through simple examples

Example 1: one query on $|0\rangle$

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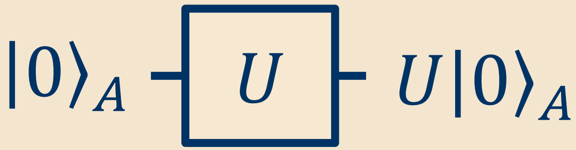
The algorithm:



$(U \leftarrow \text{Haar})$

Example 1: one query on $|0\rangle$

The algorithm: $|0\rangle_A$ —[U]— $U|0\rangle_A$ ← Fact: this is the "maximally mixed" state

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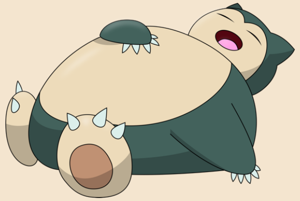

$$\sum_y |y\rangle_A {\color{red}|y\rangle_S}$$

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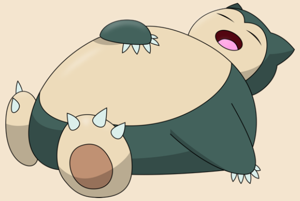

$$\sum_y |y\rangle_A |y\rangle_S$$ ← Fact: this is *also* the maximally mixed state.

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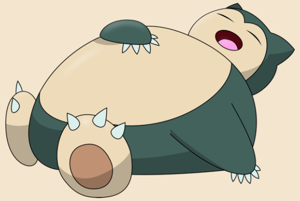

$$\sum_y |y\rangle_A |y\rangle_S$$

← Fact: this is ***also*** the maximally mixed state.

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Idea 1: entanglement with a hidden register S can simulate one query to U.

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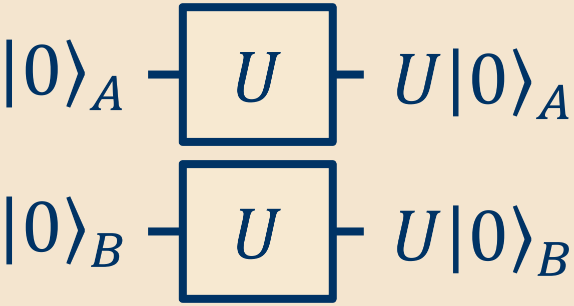

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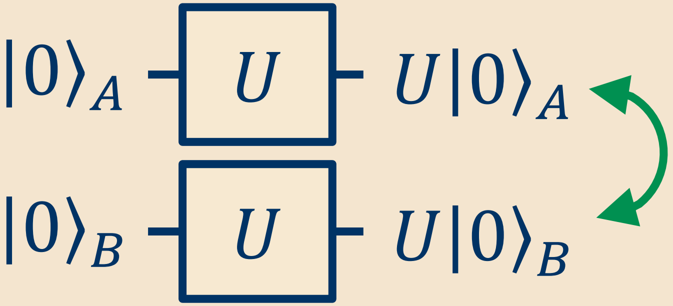

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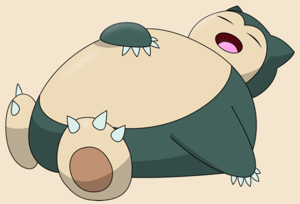

$$\sum_{y_1, y_2} |y_1\rangle_A |y_2\rangle_B |\{y_1, y_2\}\rangle_S$$

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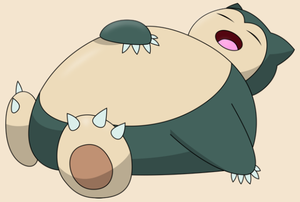

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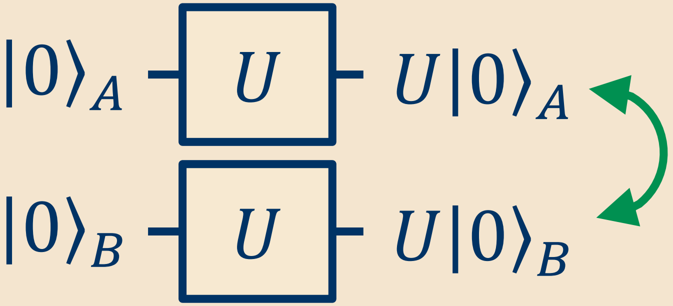

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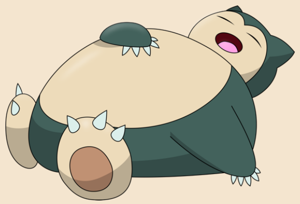

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Idea 2: use an **unordered set** to spoof "swap-symmetry".

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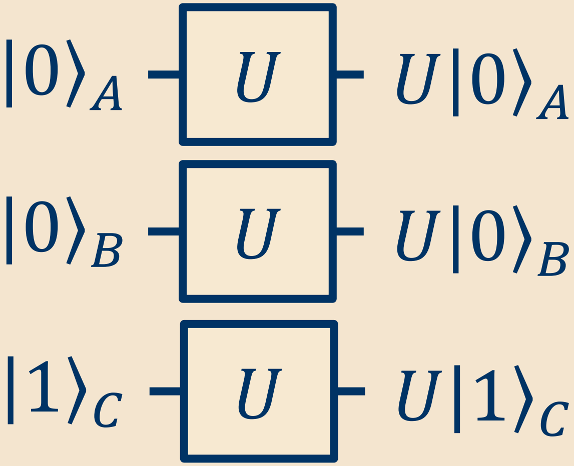

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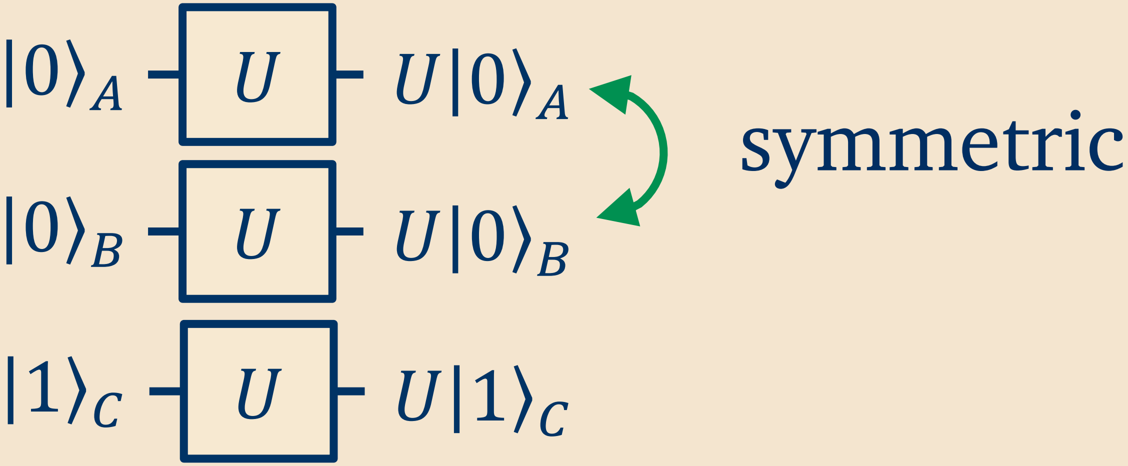

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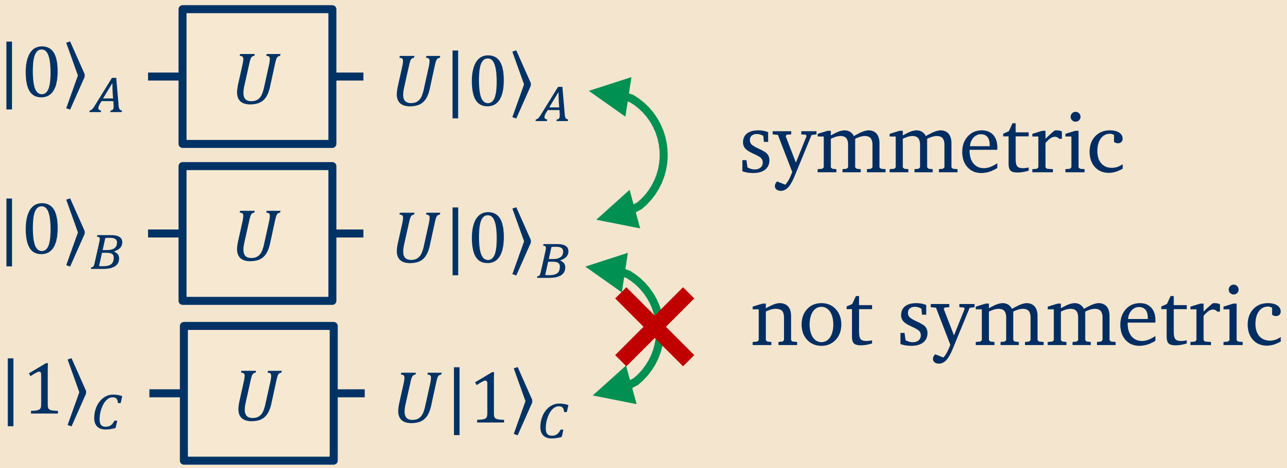

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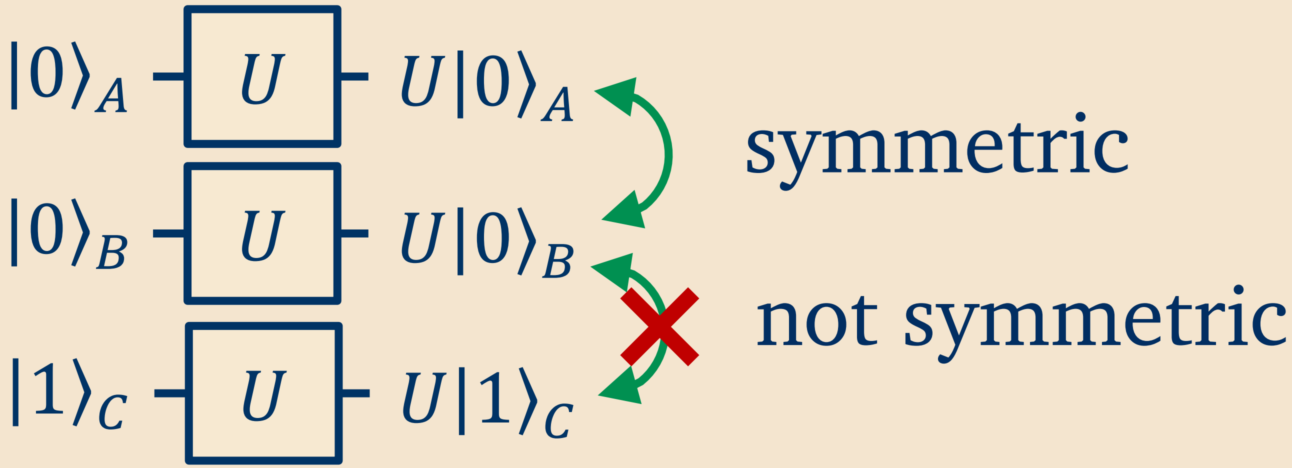

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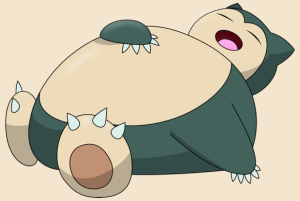

$$\sum_{y_1,y_2,y_3} |y_1\rangle_A |y_2\rangle_B |y_3\rangle_C |\{(0,y_1),(0,y_2),(1,y_3)\}\rangle_S$$

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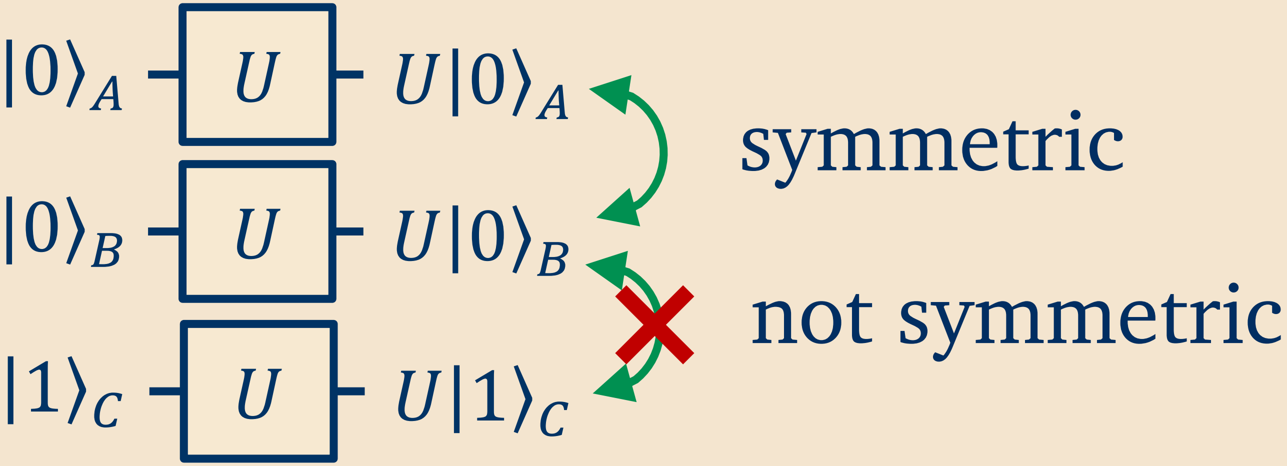

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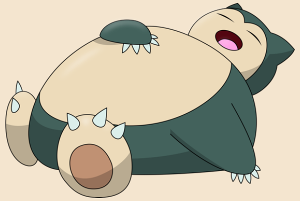

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$|1\rangle_C$ — U — $U|1\rangle_C$

$U|0\rangle_A$ ↔ $U|0\rangle_B$: symmetric

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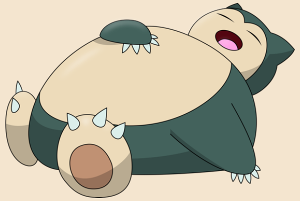

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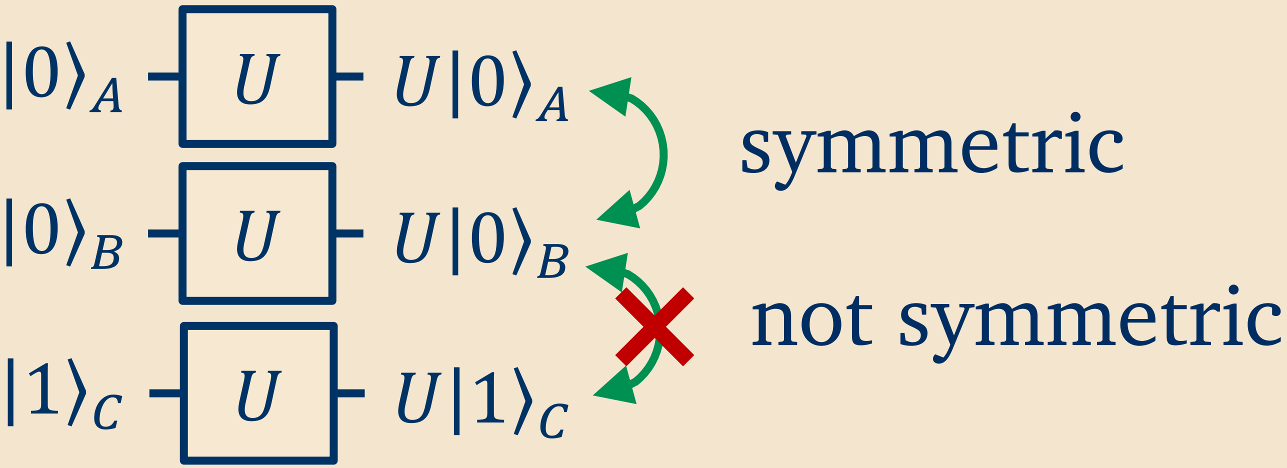

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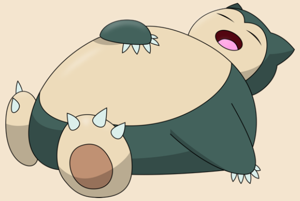

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Idea 3: use **ordered pairs** to simulate symmetry "structure"

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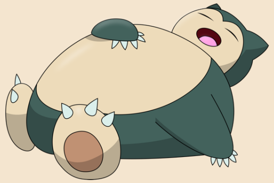

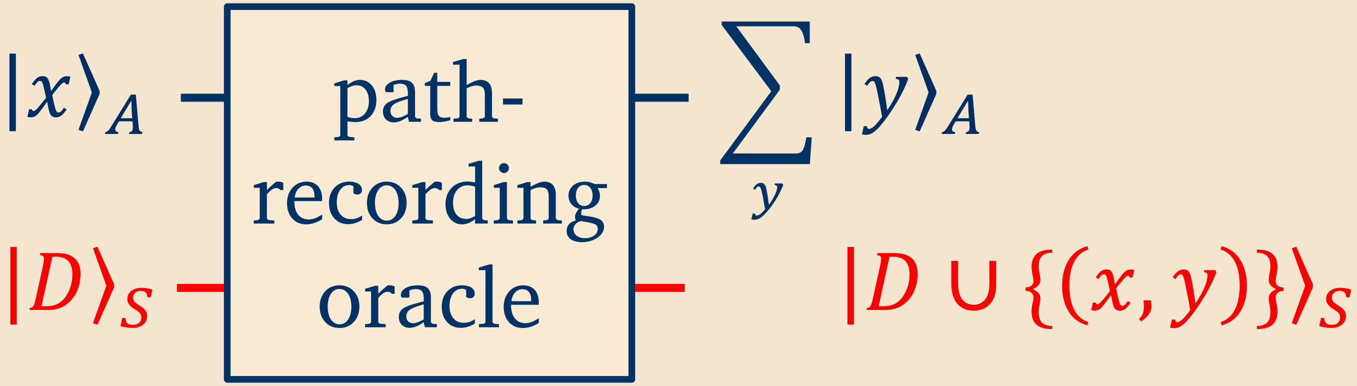

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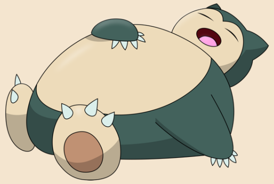

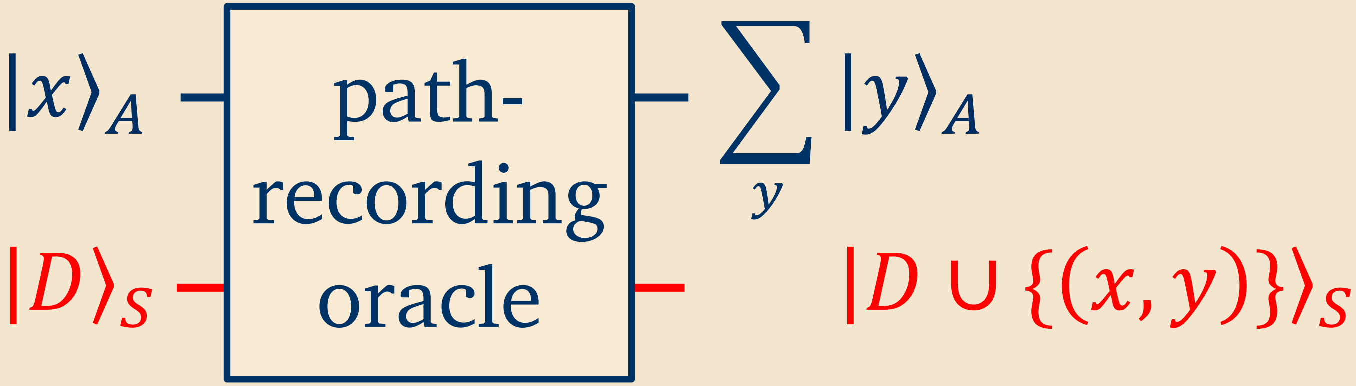

Note the similarity to classical lazy sampling:

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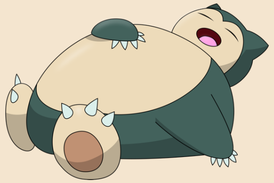

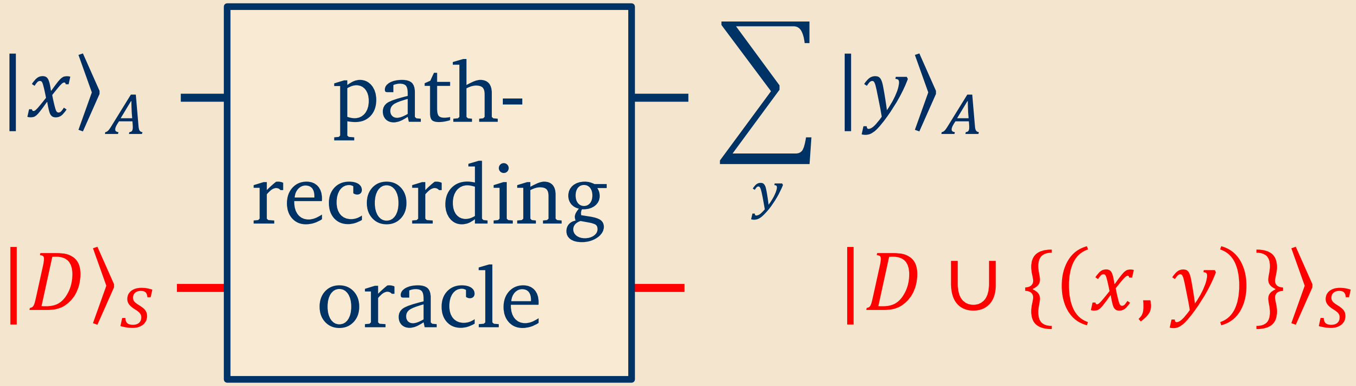

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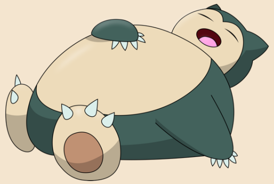

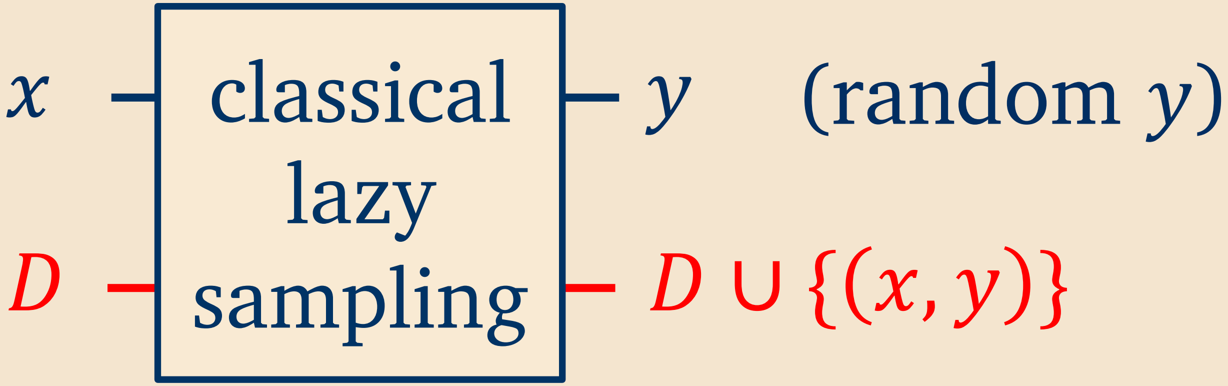

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For all efficient adversaries A,

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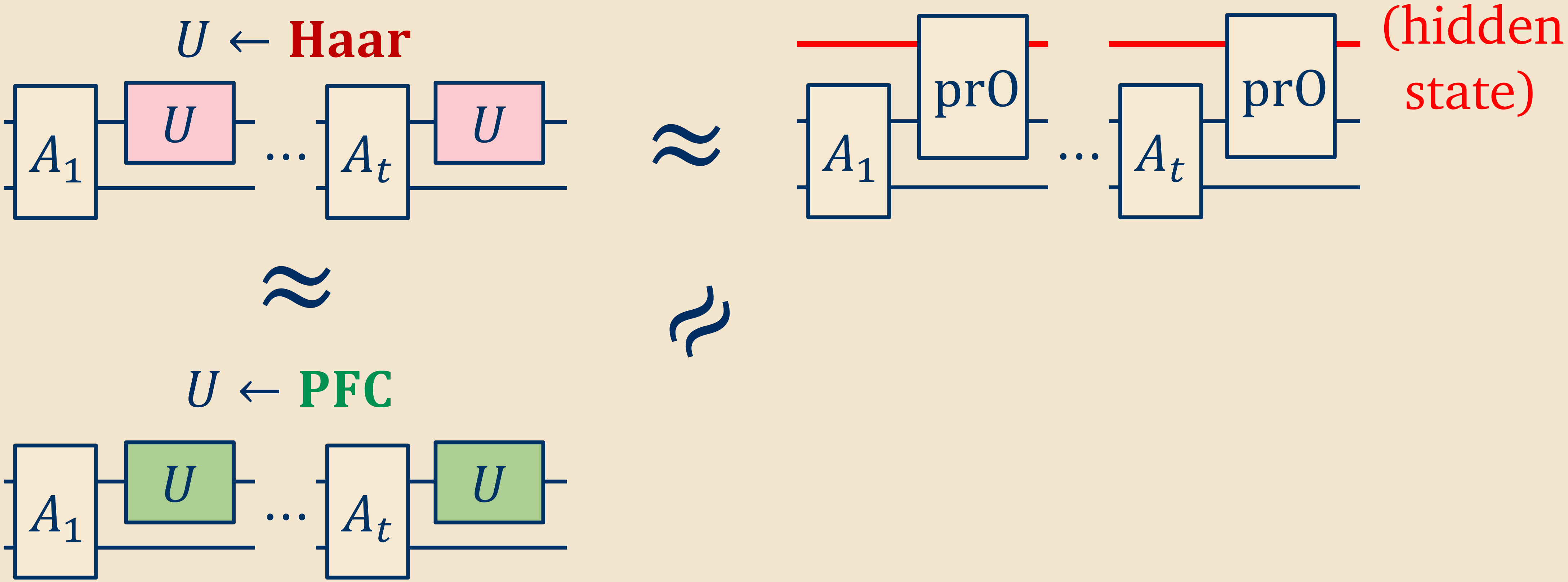

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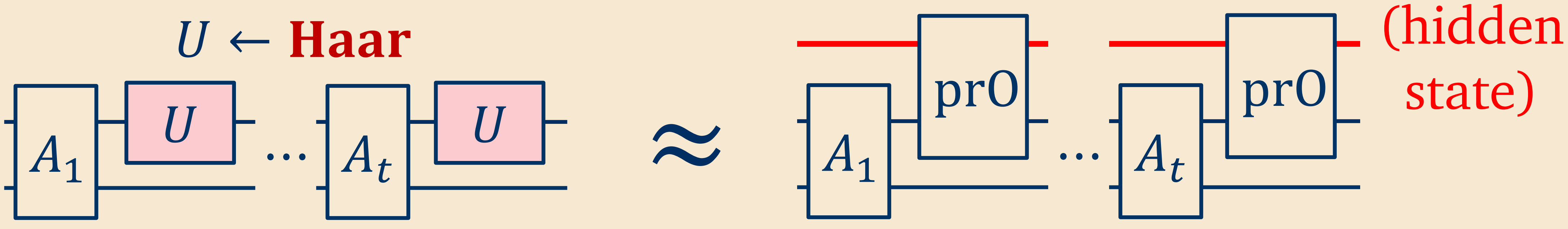

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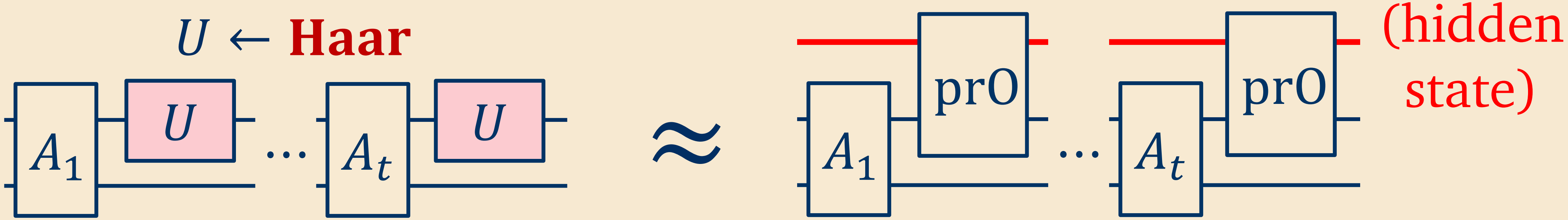

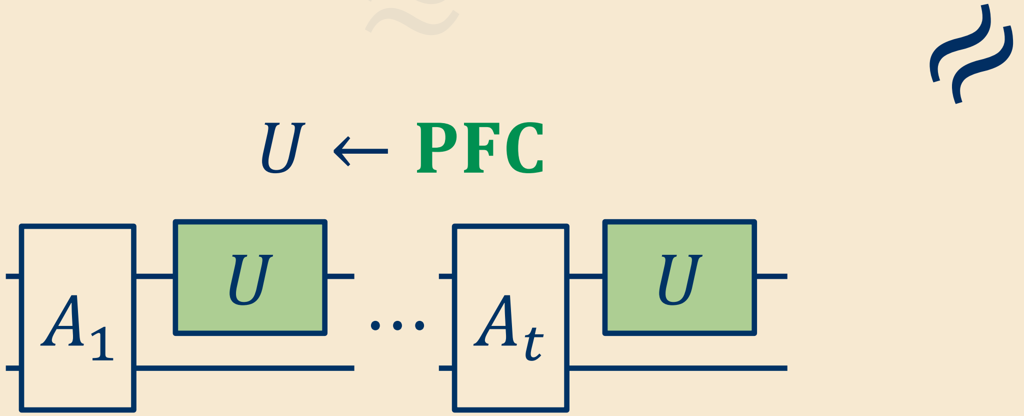

The same proof will show this!

$U \leftarrow \text{Haar}$

Hybrid 0

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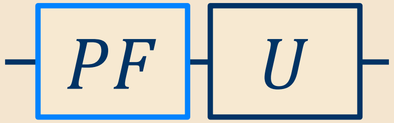

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Hybrid 0 $\equiv$ Hybrid 1

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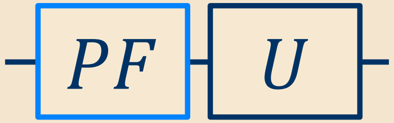

$U \leftarrow \text{Haar}$

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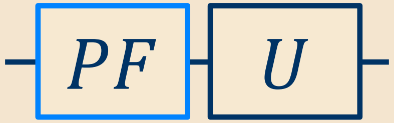
Hybrid 0 $\equiv$ Hybrid 1

Step 1: insert random permutation P random ± 1 diagonal F.

$$P = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \qquad F = \begin{pmatrix} +1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

$U \leftarrow \text{Haar}$

Hybrid 0 $\equiv$

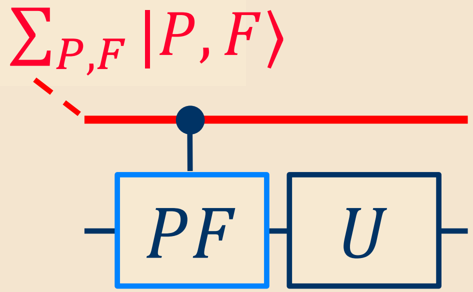


$U \leftarrow \text{Haar}$

$P \leftarrow S_N$

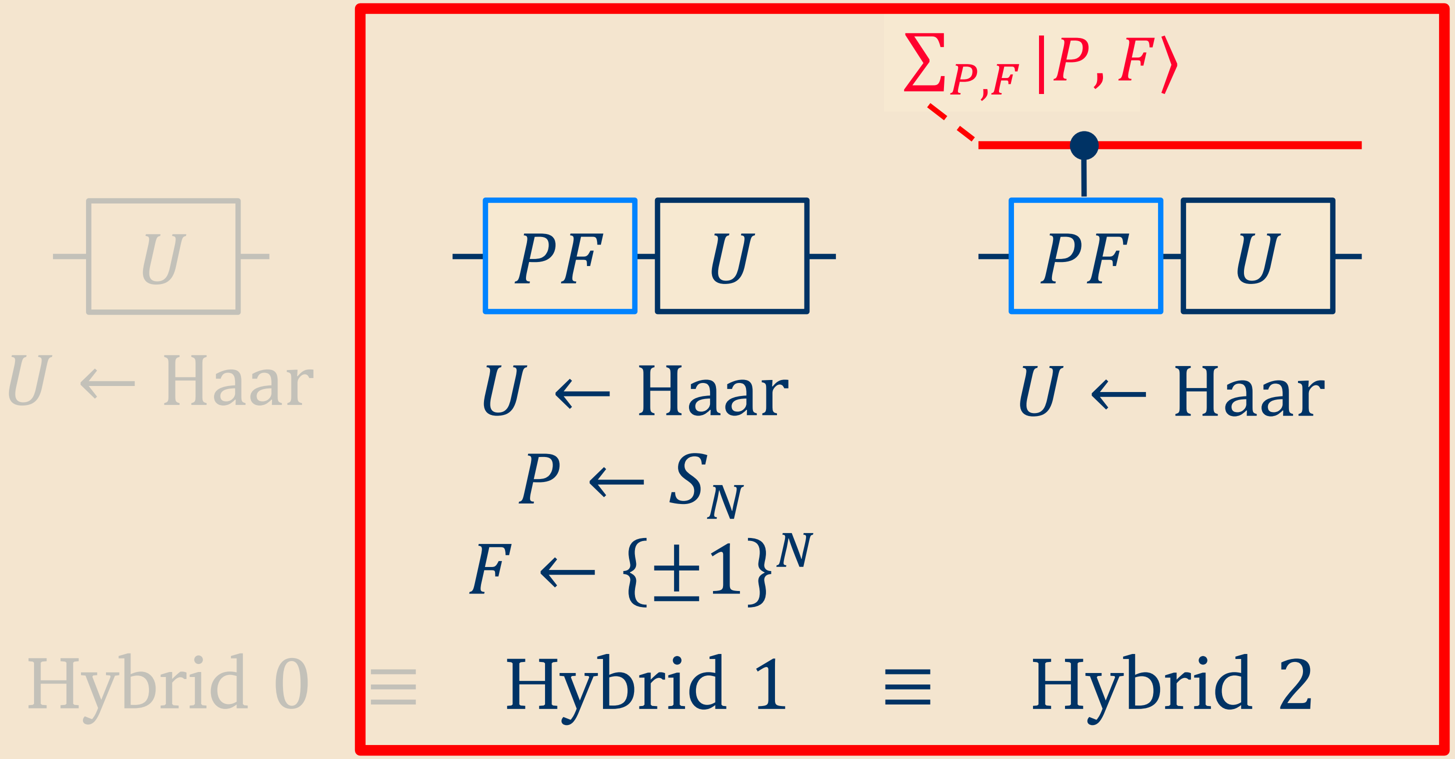
$F \leftarrow \{\pm 1\}^N$

Hybrid 1 $\equiv$



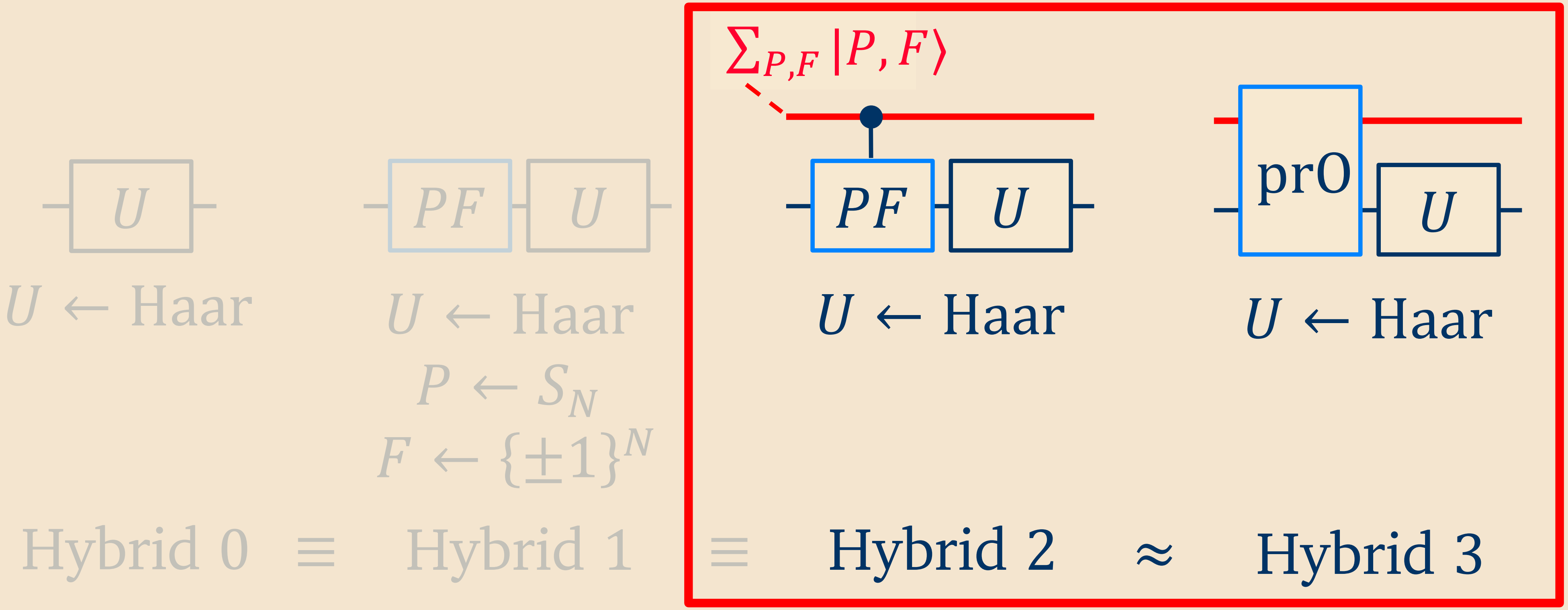
$U \leftarrow \text{Haar}$

Hybrid 2

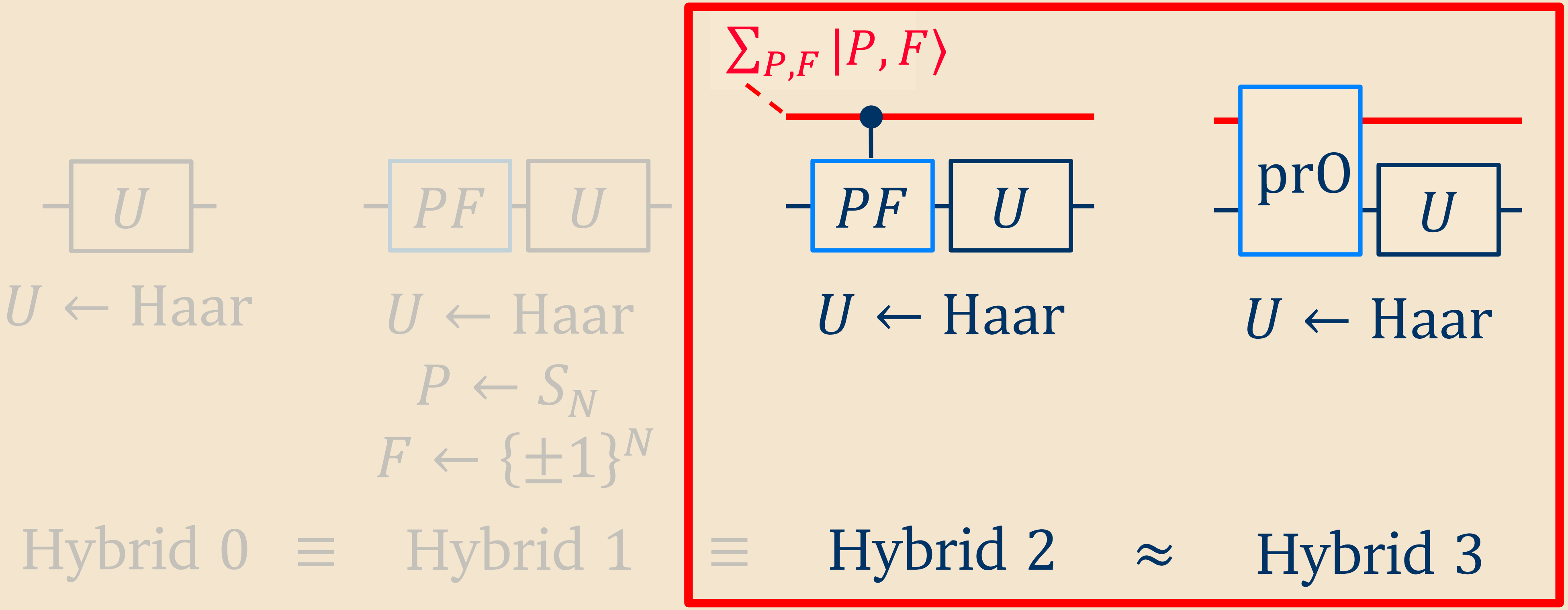


Step 2: replace random P, F with a purification.

- Initialize external/ancilla system to $\sum_{P,F} |P,F\rangle$
- On each query, apply $P \cdot F$ **controlled** on $|P,F\rangle$



U
U ← Haar
PF
U
U ← Haar
P ← S_N
F ← {±1}^N
Hybrid 0 ≡ Hybrid 1 ≡ Hybrid 2 ≈ Hybrid 3
∑_{P,F} |P, F⟩
PF
U
U ← Haar
prO
U
U ← Haar



Step 3: Key idea: analyze ctl-PF in a different basis.

Let's see how this works…

(1) ctl-PF: $|x\rangle \otimes |P, F\rangle \mapsto (-1)^{F(x)}|P(x)\rangle \otimes |P, F\rangle$

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Definition: for $D = \{(x_1, y_1), \ldots, (x_t, y_t)\}$,

$$|\Phi_D\rangle := \sum_{P,F} (-1)^{F(x_1)+\cdots+F(x_t)} \cdot \delta_{P(x_1)=y_1} \cdots \delta_{P(x_t)=y_t} \, |P, F\rangle$$

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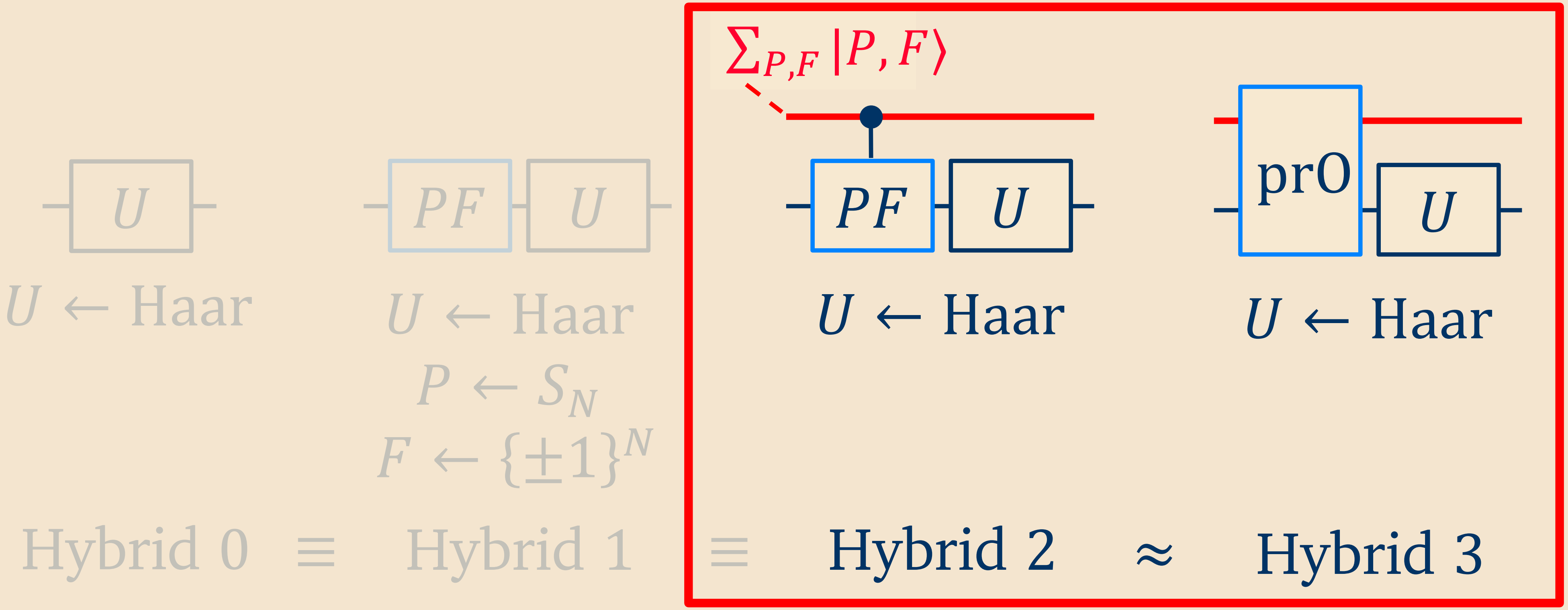
pr0: $|x\rangle \otimes |D\rangle \mapsto \sum_y |y\rangle \otimes |D \cup \{(x,y)\}\rangle$



Step 3: For any $D = \{(x_1, y_1), \ldots, (x_t, y_t)\}$ can define $|\Phi_D\rangle$ s.t.

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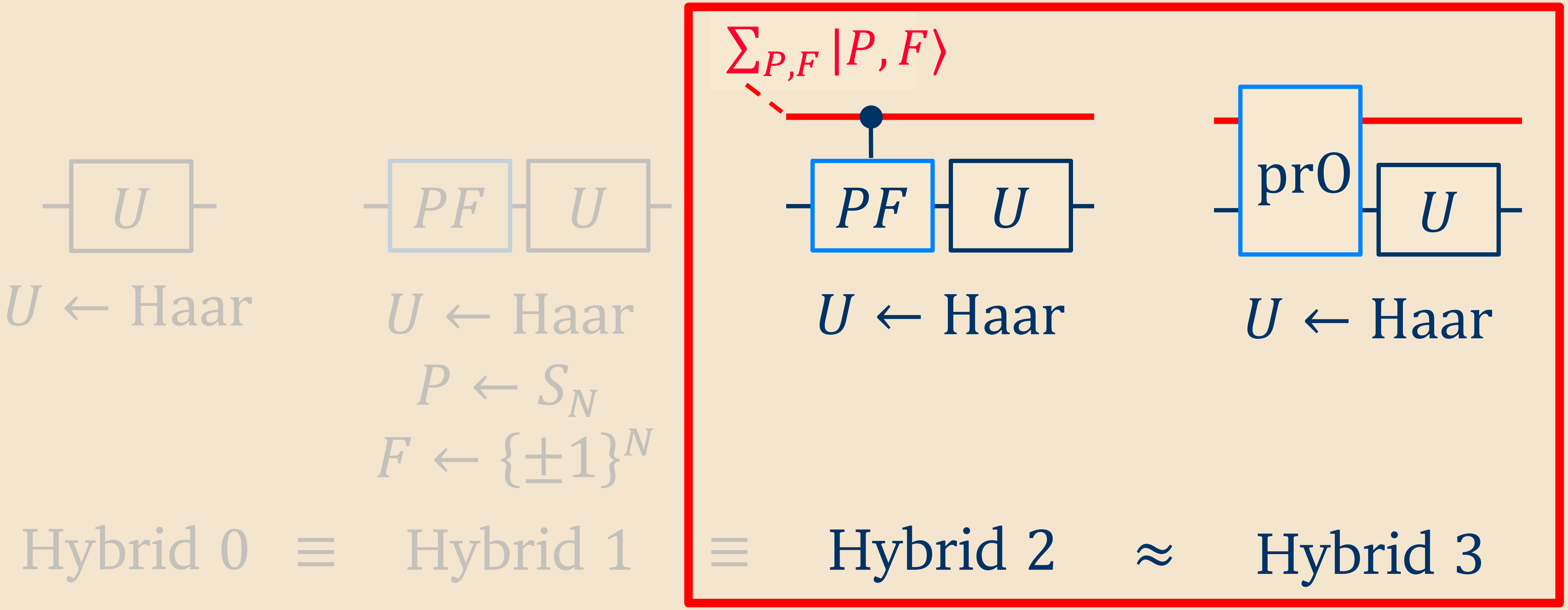

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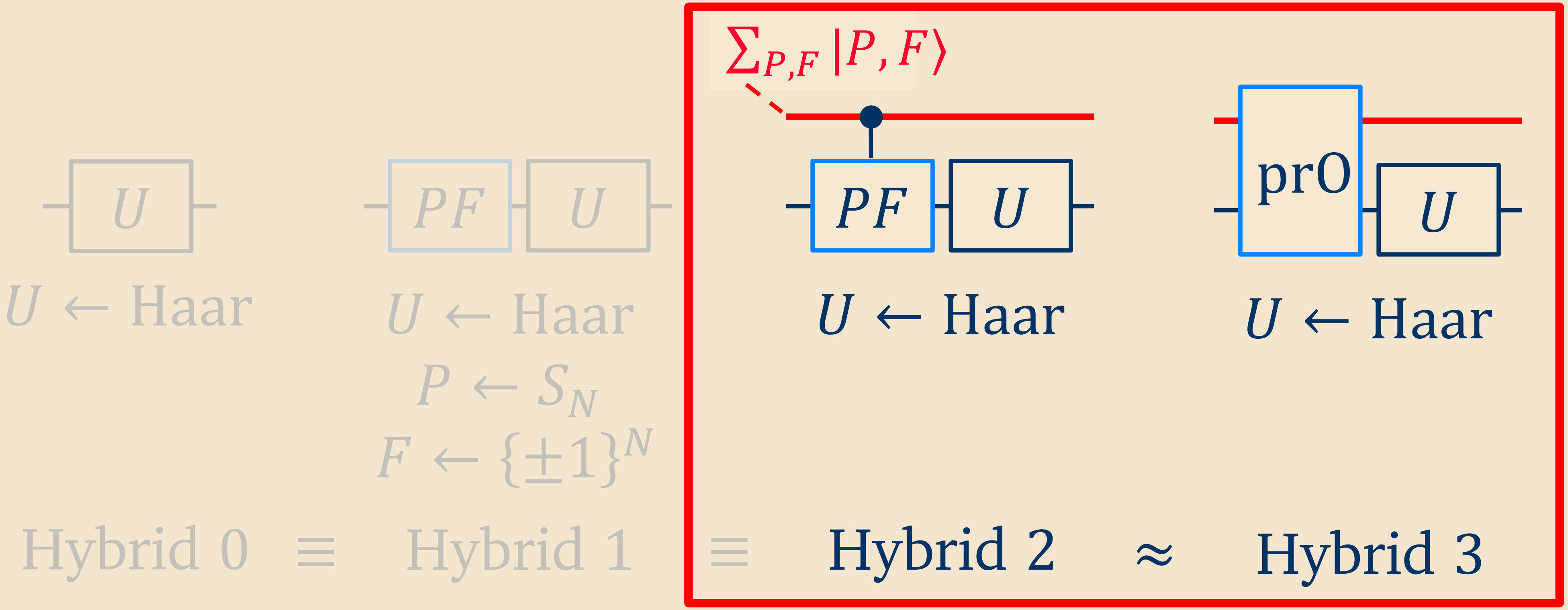
- **Intuition:** ctl-PF behaves like prO, up to relabeling $|\Phi_D\rangle \mapsto |D\rangle$



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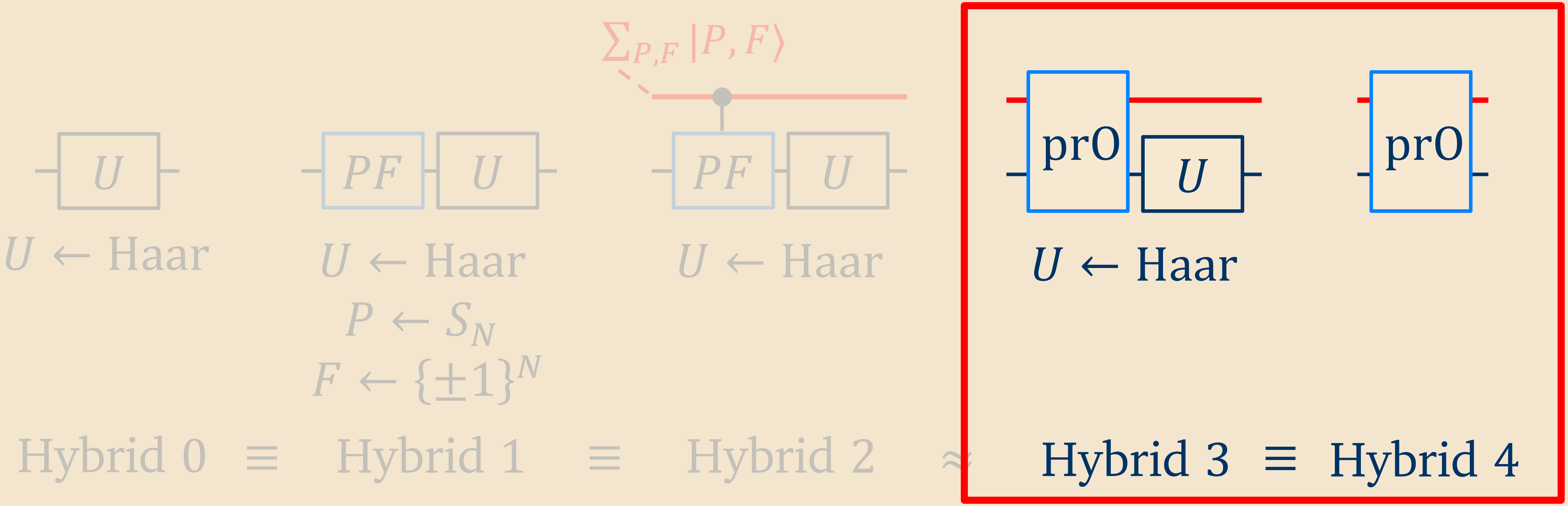
- **Intuition:** ctl-PF behaves like prO, up to relabeling $|\Phi_D\rangle \mapsto |D\rangle$
- Actually, $\{|\Phi_D\rangle\}_D$ aren't fully orthogonal. But composing with $U \leftarrow$ (2-design) makes the "non-orthogonal" ones hard to find.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PF
U
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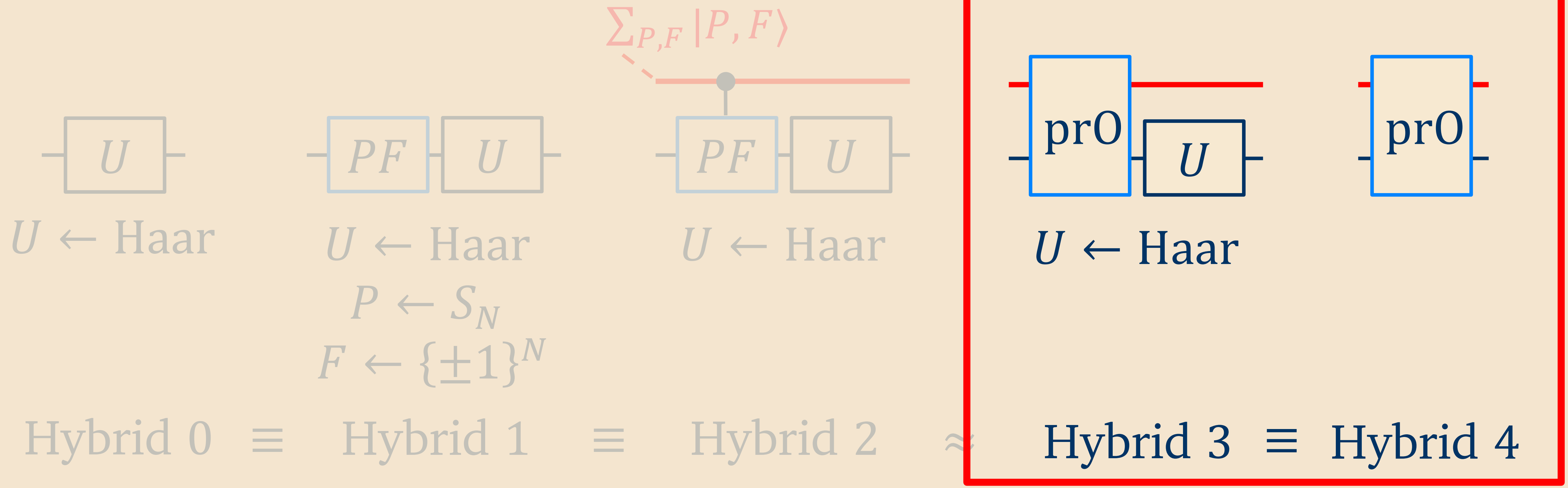
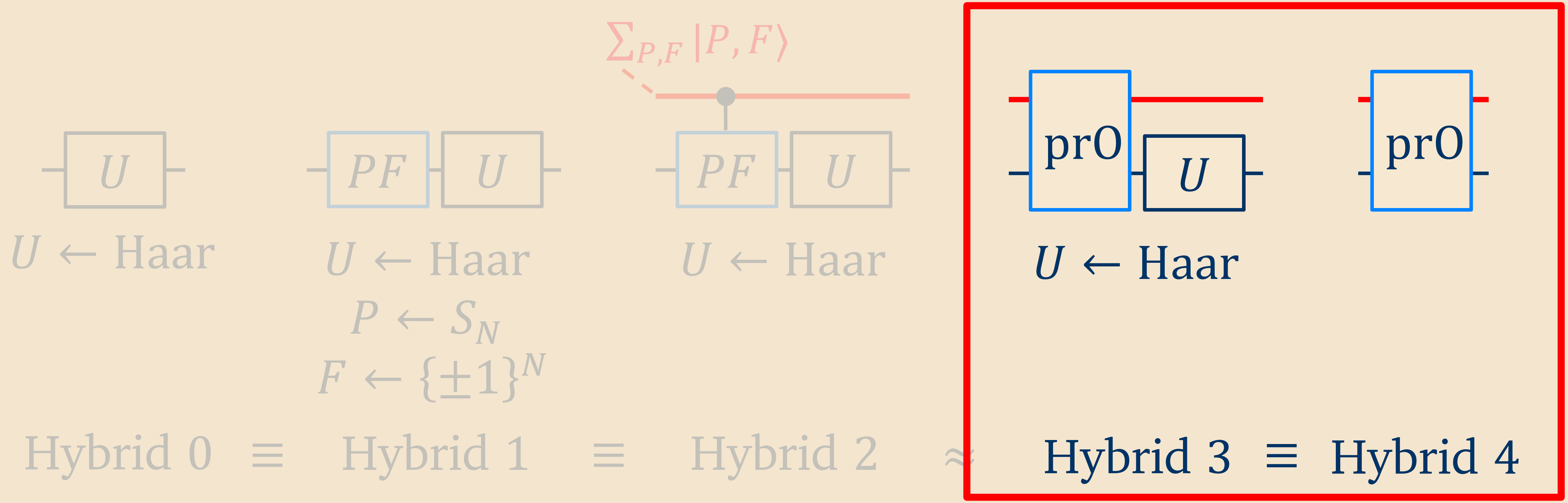
Hybrid 0 $\equiv$ Hybrid 1 $\equiv$ Hybrid 2 $\approx$ Hybrid 3 $\equiv$ Hybrid 4



Step 4: Turns out prO has the following unitary invariance property:



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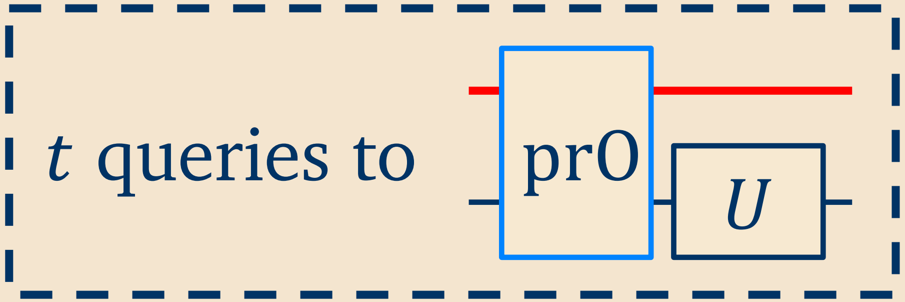

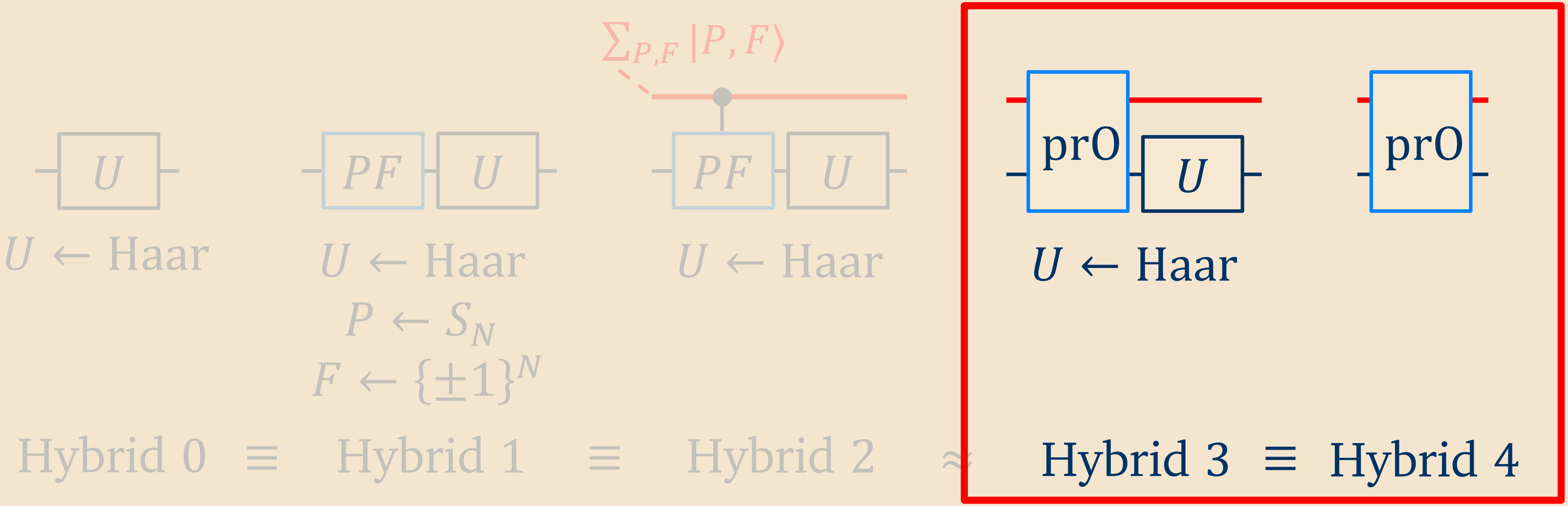

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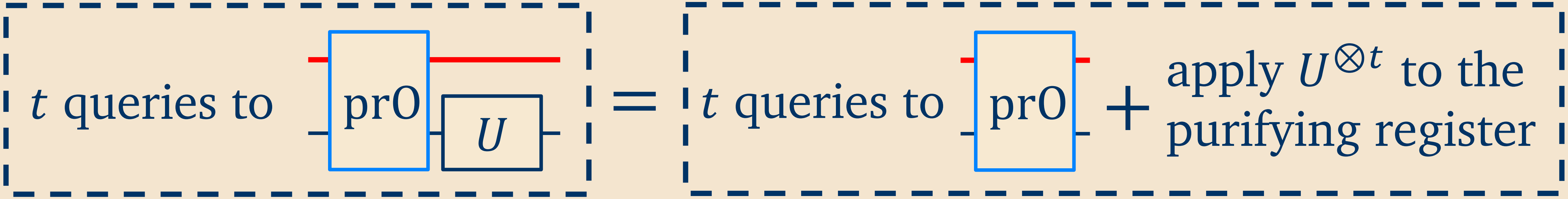

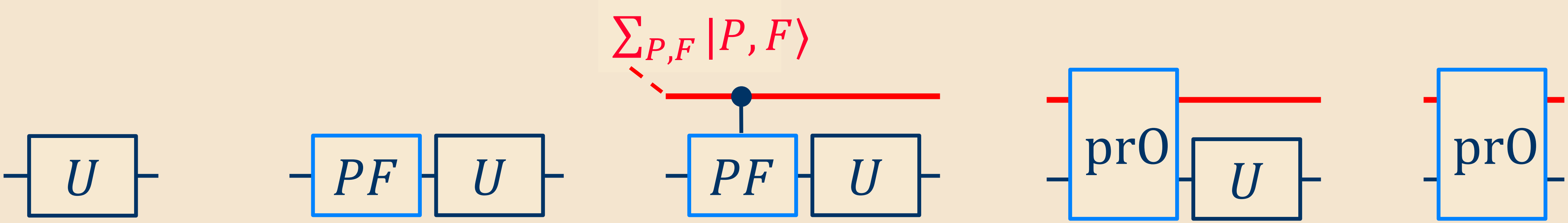

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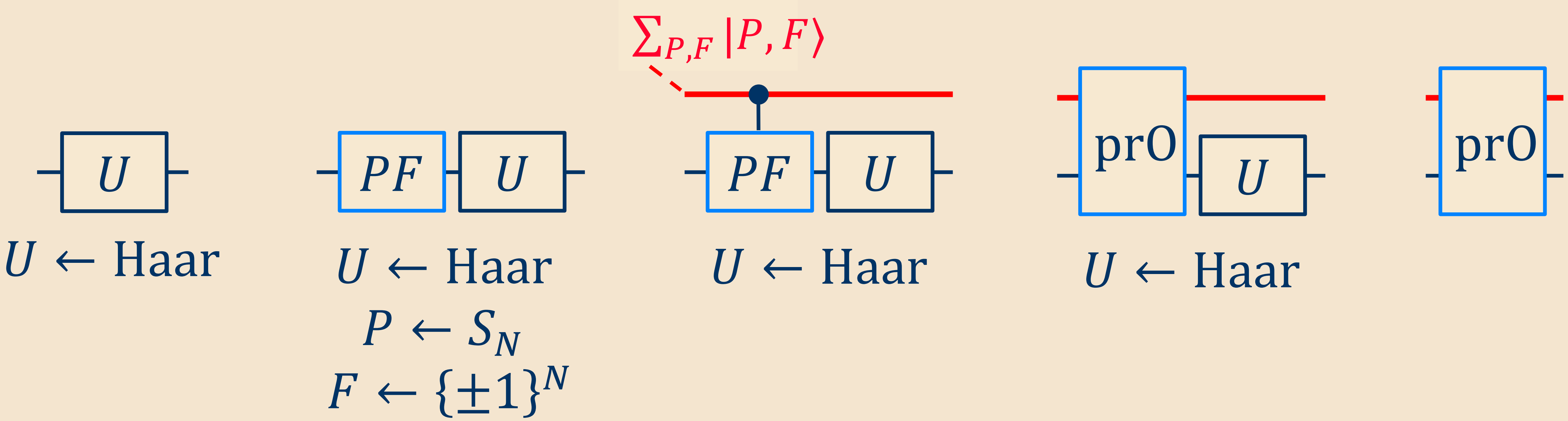

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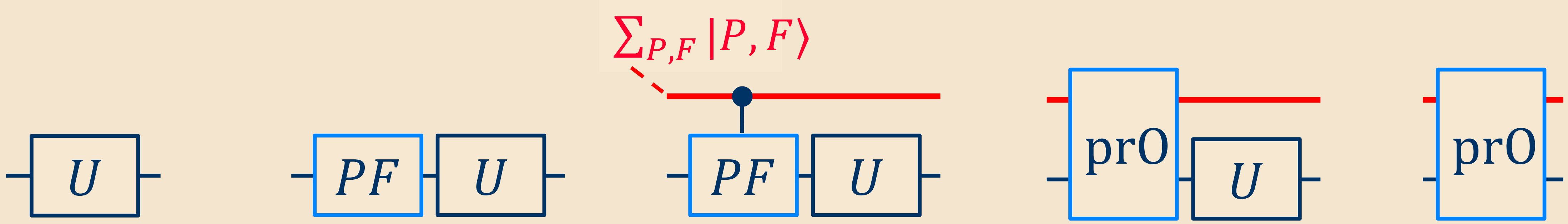
The PRU proof: Hybrid 2 $\approx$ Hybrid 4 holds for any **2-design**.



$U \leftarrow$ Haar | $U \leftarrow$ Haar, $P \leftarrow S_N$, $F \leftarrow \{\pm 1\}^N$ | $U \leftarrow$ Haar | $U \leftarrow$ Haar

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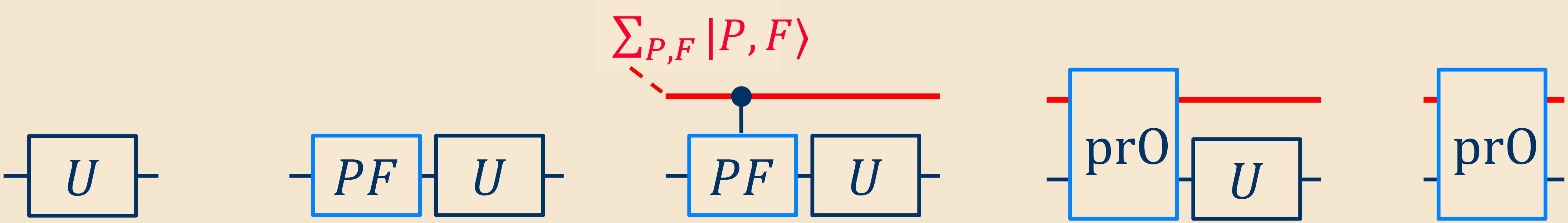

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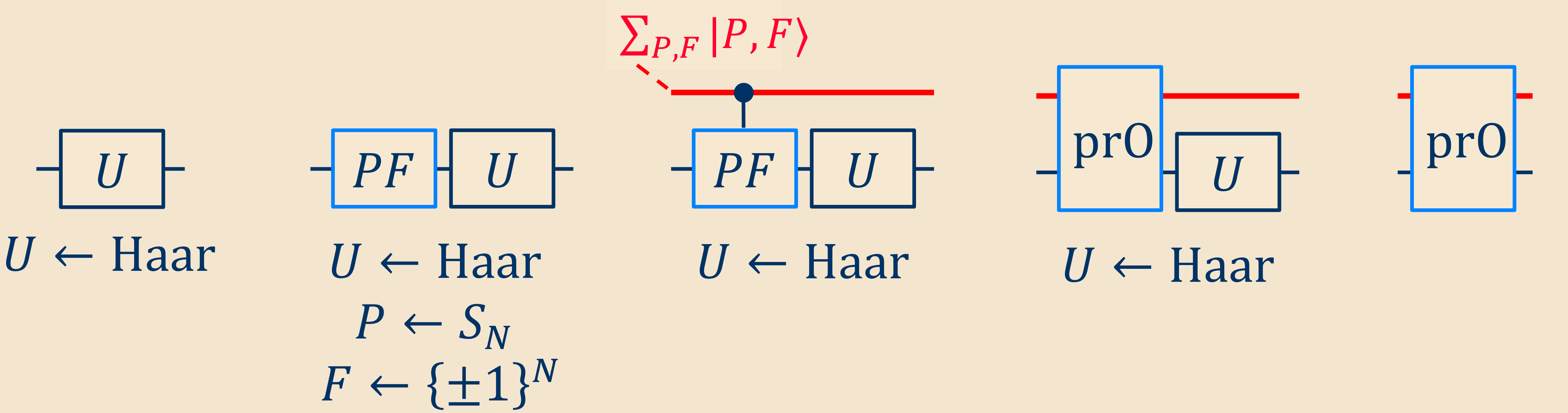

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Finally, replace P and F with pseudorandom.

Rest of this talk

- Lazy sampling of a random function
- Lazy sampling of a random unitary
- Proving correctness + PRUs exist
- **Applications**

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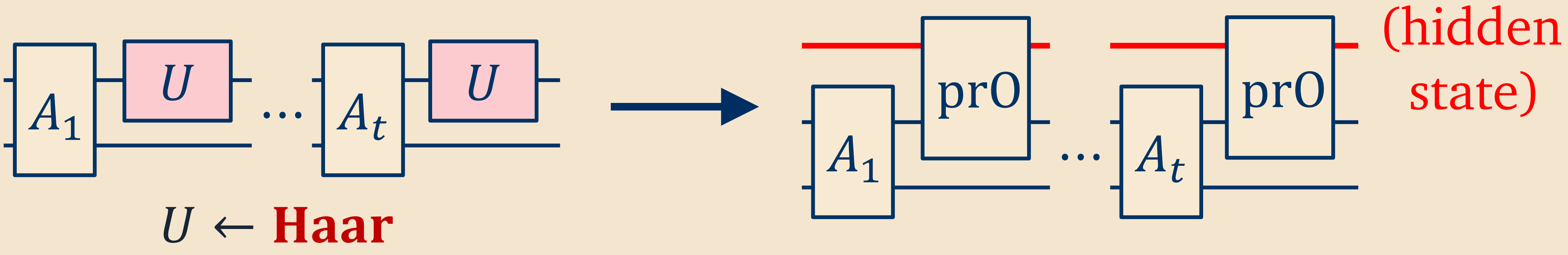

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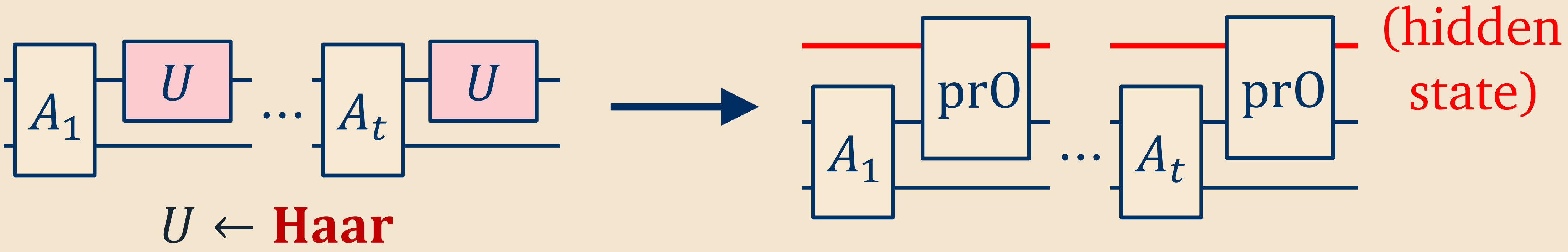

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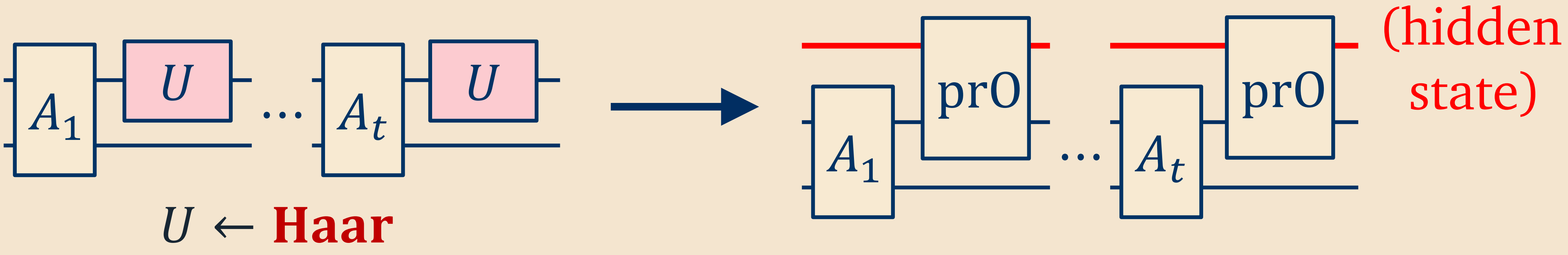

Many statements about Haar-random U can be reduced to simple claims about this data structure

- [**M**H24]: elementary proof of [SHH24] gluing lemma
- [S**M**LBH25]: existence of low-depth PRUs

Let's see an example.

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If U_1 and U_2 overlap on $|B| = \omega(\log n)$ qubits, then

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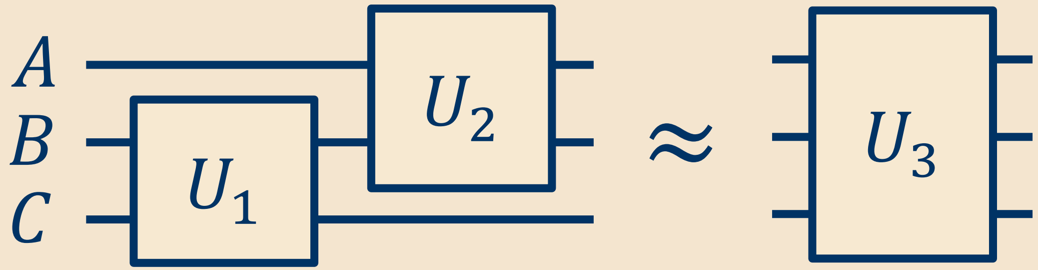

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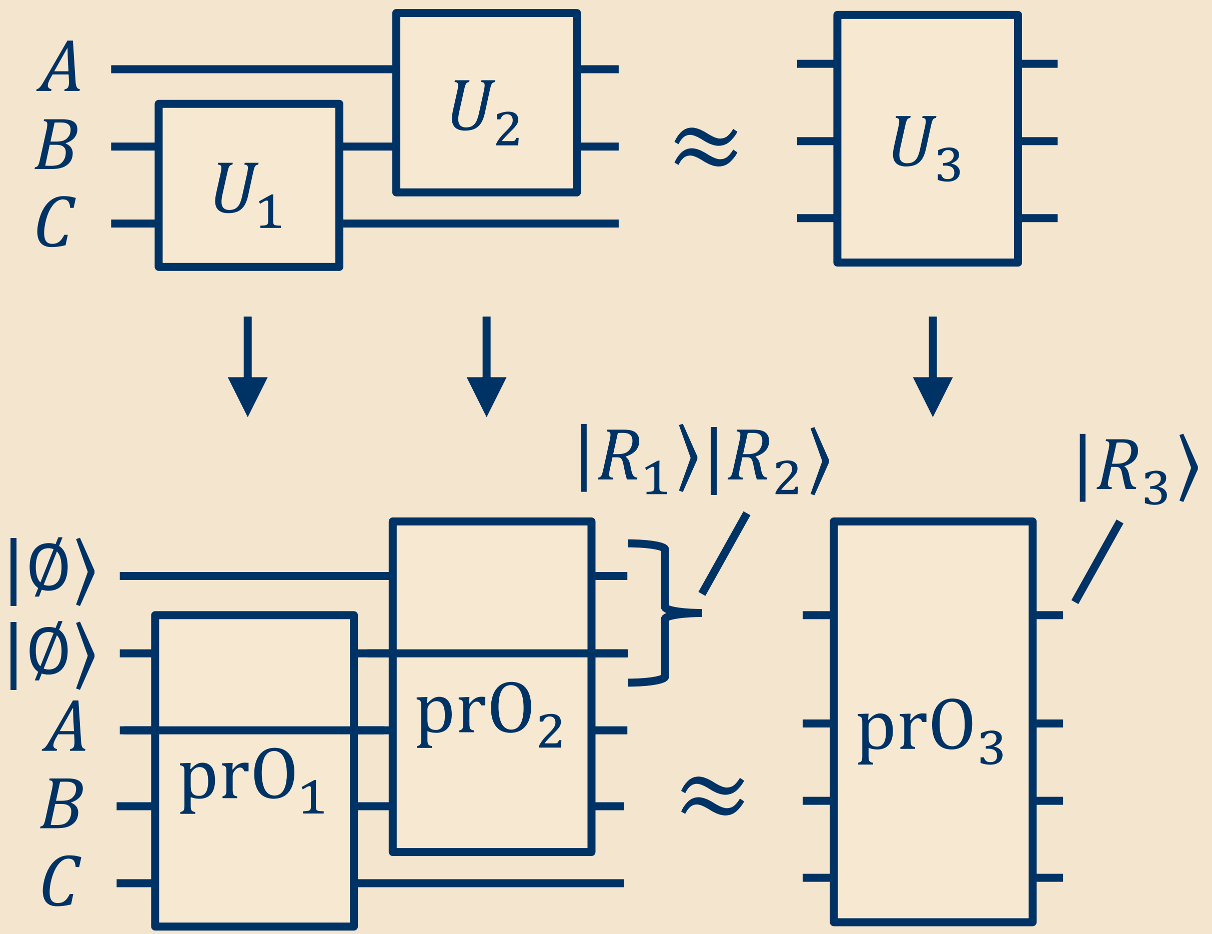
New proof: combinatorial claim about path-recording oracle.



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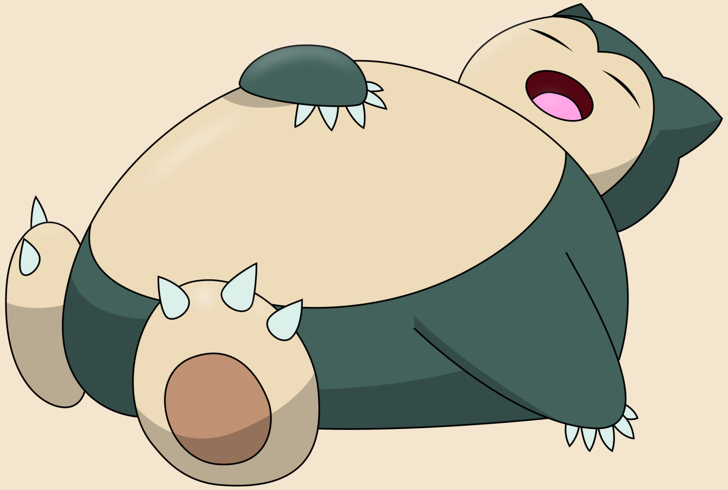

Thanks!