Pseudorandom Unitaries and Compressed Purifications

Fermi Ma

joint work with Hsin-Yuan Huang

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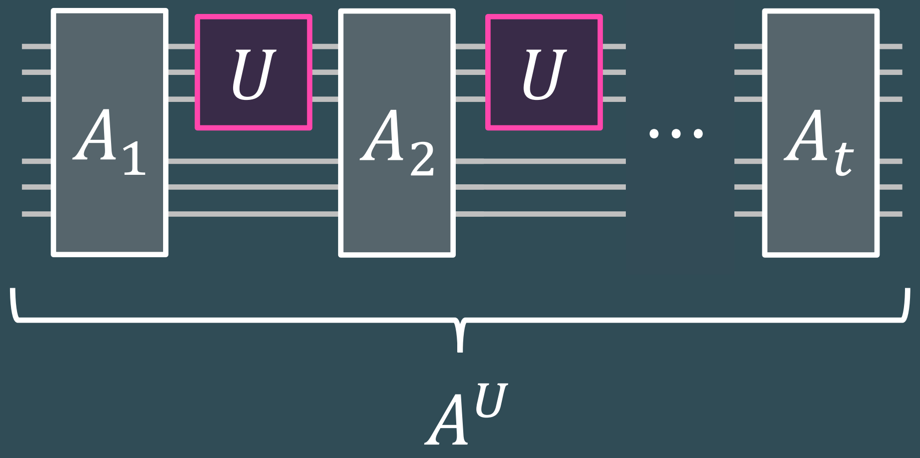

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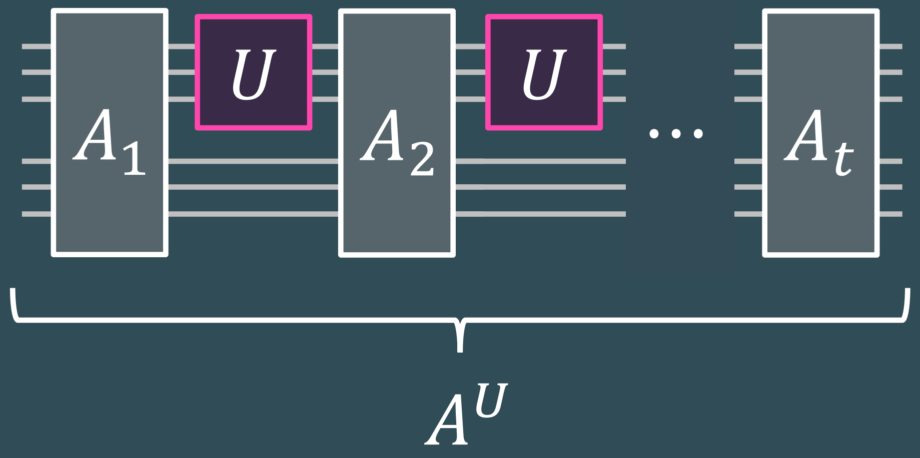

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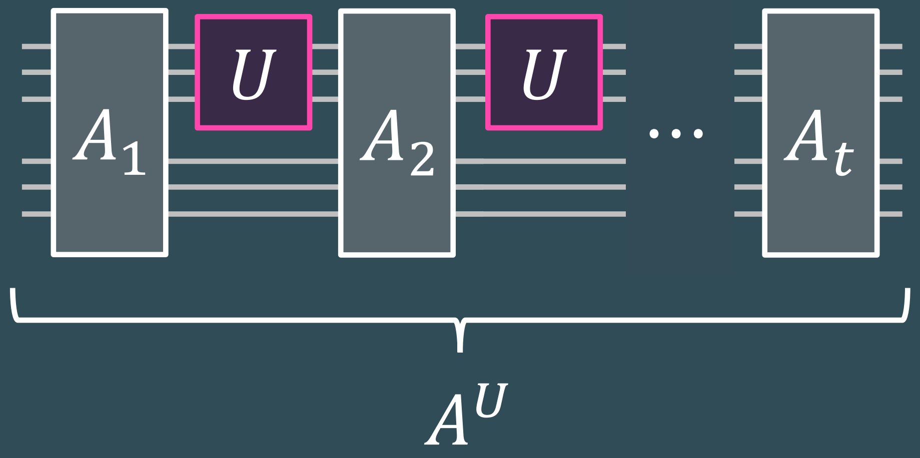

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$\{U_k\}$ is a **pseudorandom unitary (PRU)**.

(as defined by JLS18)

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- New perspectives on random unitaries

Open question: do PRUs exist? (under cryptographic assumptions)

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Prior work: PRUs secure against **restricted** adversaries [LQSYZ23,AGKL23,BM24,MPSY24,CBBDHX24,...]

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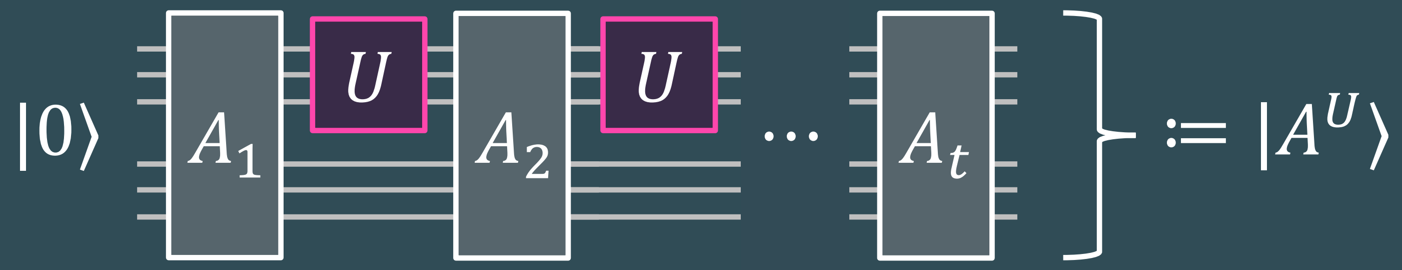

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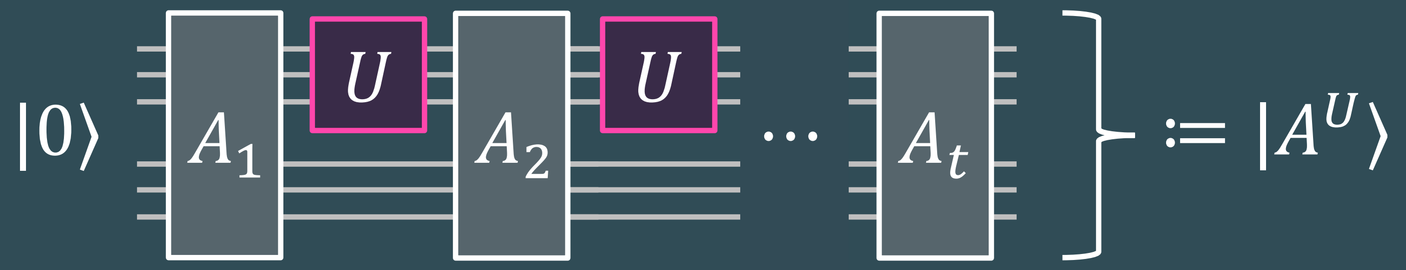

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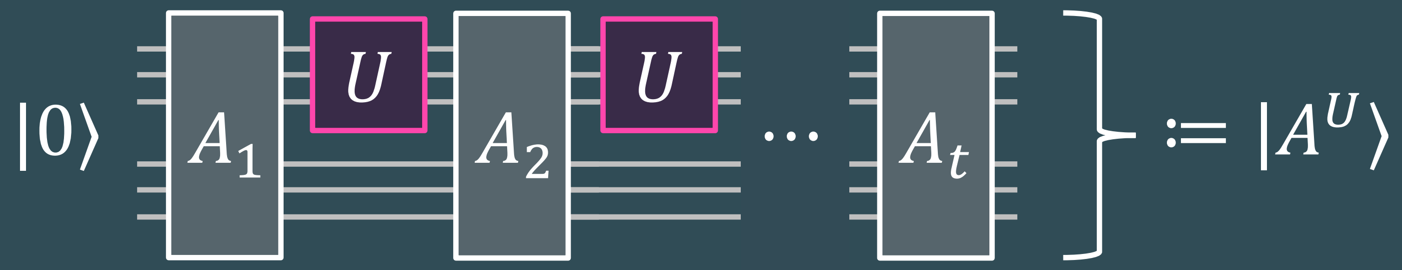

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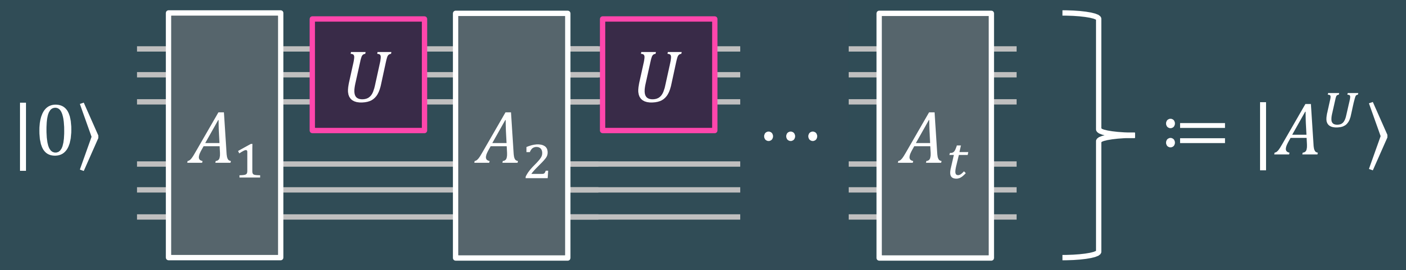

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where the sum is over all partitions λ of q (Collins 2003). Here χ^{λ} is the character of S_q corresponding to the partition λ and s is the Schur polynomial of λ, so that $s_{\lambda d}(1)$ is the dimension of the representation of U_d corresponding to λ.

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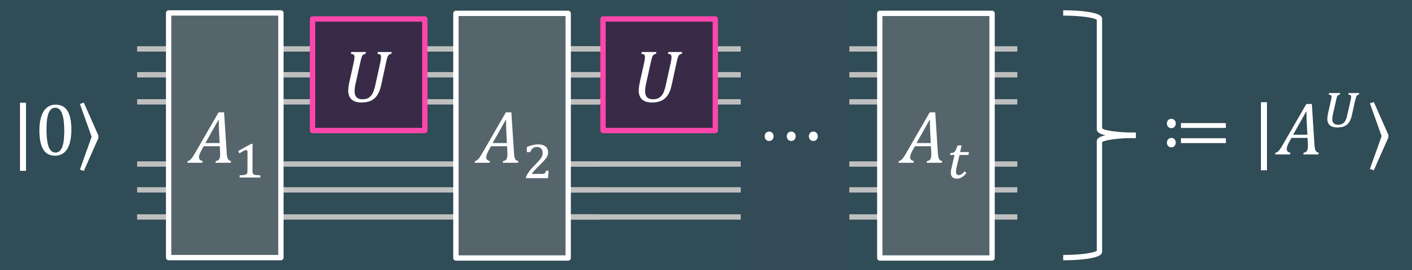

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... but even this only gives you *entries* of the RHS!

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(this talk)

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Philosophy: try to ***compress*** the purification.

Let’s do a simple example.

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$$|\phi_x\rangle = \sum_{f\in\{0,1\}^N} (-1)^{f\cdot e_x}|f\rangle = H^{\otimes N}|e_x\rangle$$

So $|\phi_x\rangle$'s are **orthogonal** and we can map $|\phi_x\rangle \mapsto |x\rangle$

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$$\sum_{x\in[N]}|x\rangle|x\rangle$$

Since this is a purification of ρ, this means ρ is maximally mixed!

Rest of today: [MH24] PRU proof on the blackboard

Preliminary draft of the paper:

fermima.com/pru.pdf