How to Construct Random Unitaries

Fermi Ma

(joint w/ Hsin-Yuan Huang)

Haar measure: unitarily invariant measure on $\mathrm{SU}(2^n)$

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Fundamental to understanding quantum phenomena:

- information scrambling
- entanglement
- quantum learning algorithms
- quantum crypto
- random quantum circuits
- unitary complexity
- quantum error correction

…

Challenge:

Haar-random unitaries are exponentially complex

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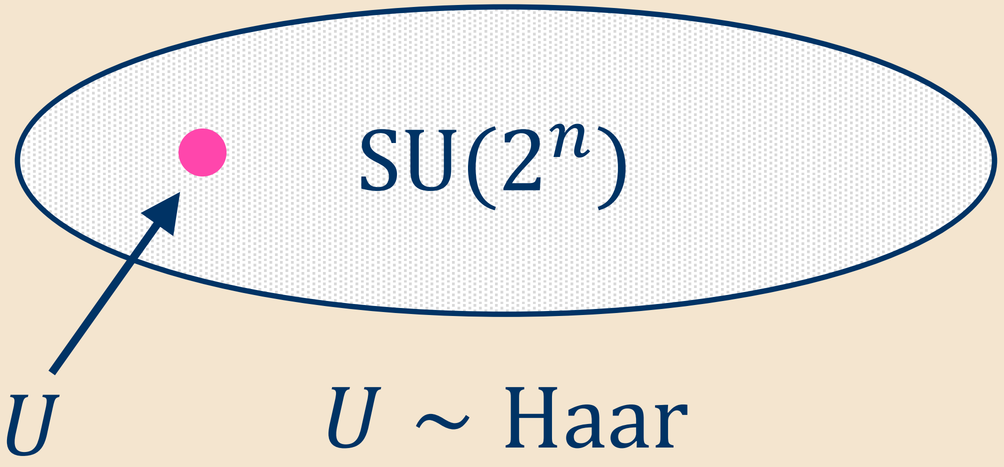

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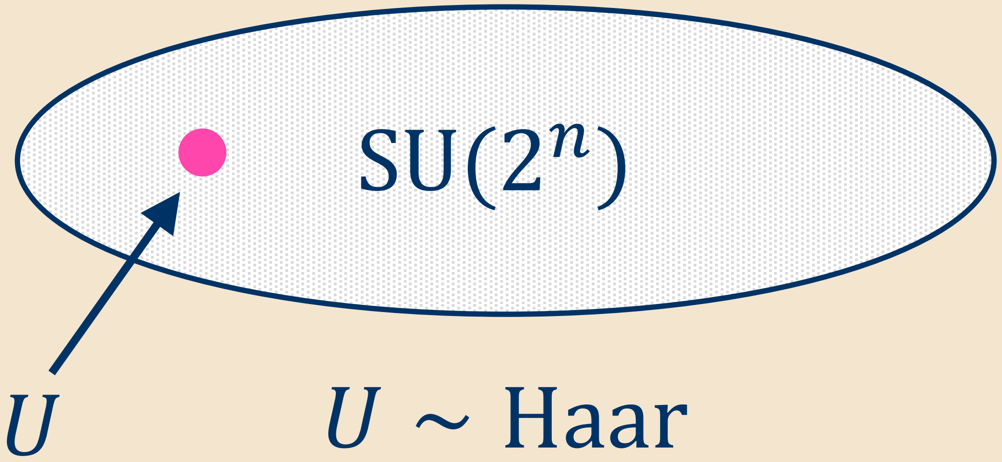

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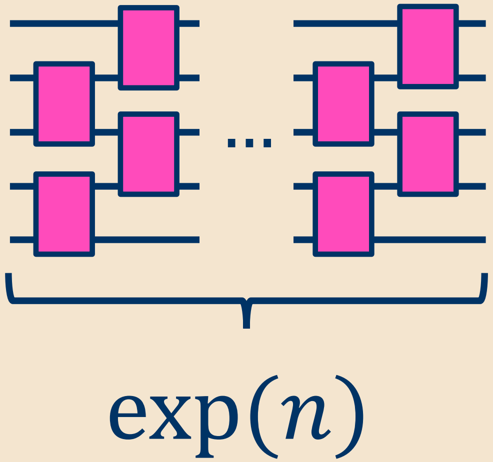

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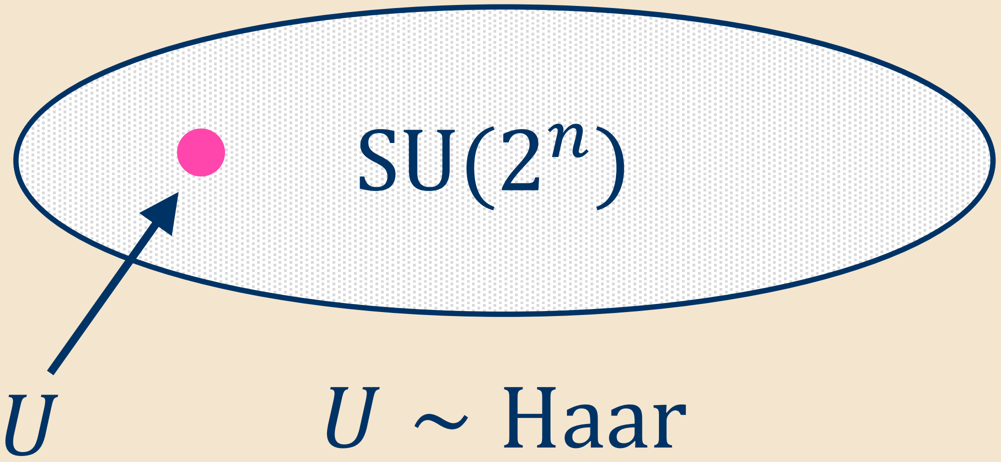

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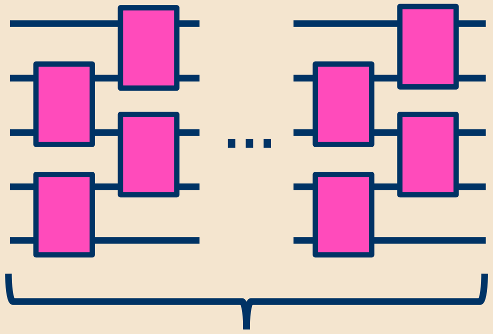

$\exp(n)$

This makes them impractical for most applications!

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$\approx$

Haar random

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Classical analogue: pseudorandom functions (PRFs) or pseudorandom permutations (PRPs)

Open question: provably-secure PRUs

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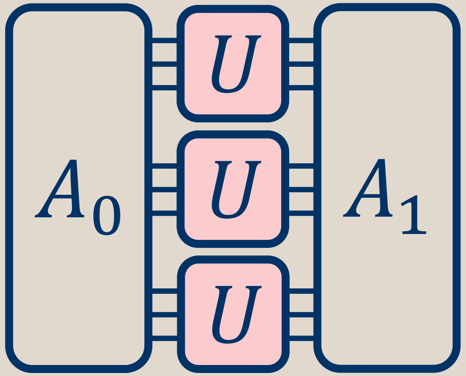

non-adaptive distinguisher

This work: PRUs exist

(if one-way functions exist)

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Same construction as [MPSY24]:

PRP $\times$ PRF $\times$ C

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- efficient simulation of Haar-random unitaries

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New technique: the path-recording oracle

- efficient simulation of Haar-random unitaries
- only uses basic quantum info (purification)

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But for this talk, I'll focus on the weaker notion.

Rest of this talk

- Warmup: simulating a random function
- This work: simulating a random unitary
- Analyze PRUs

Warmup: simulating a random *function*

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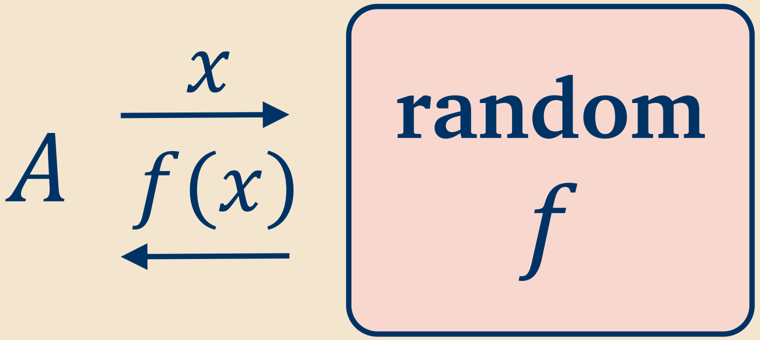

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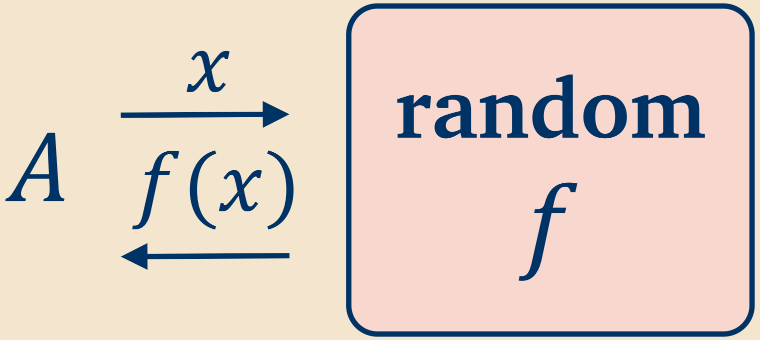

efficient simulation

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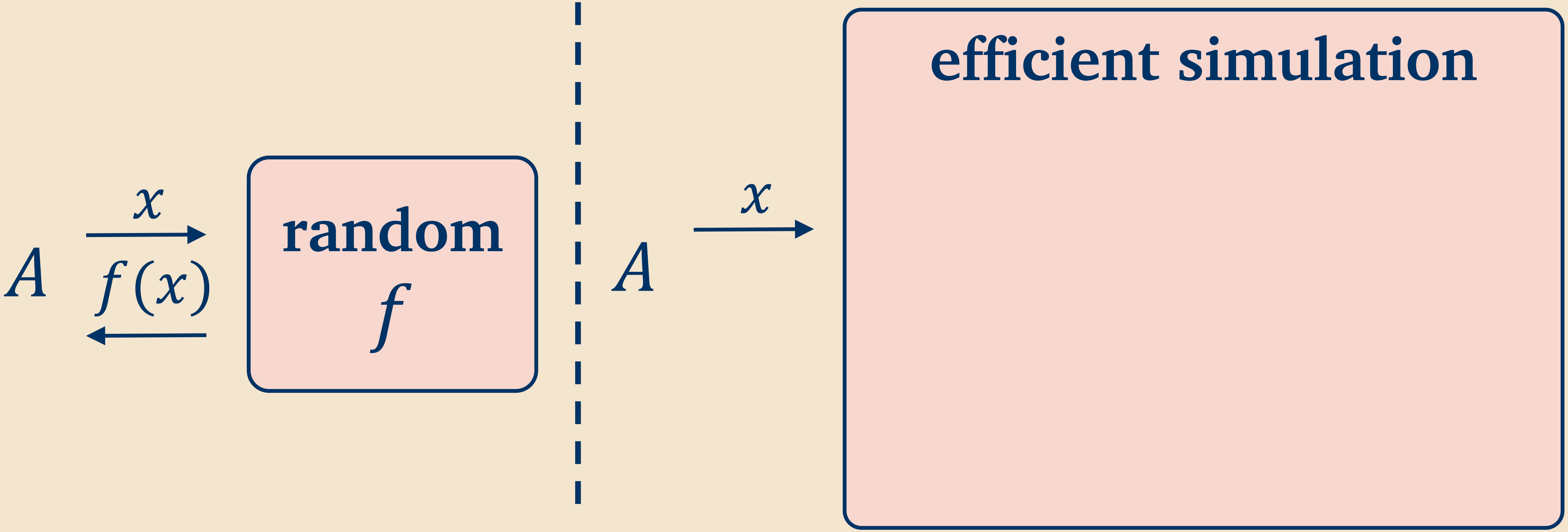

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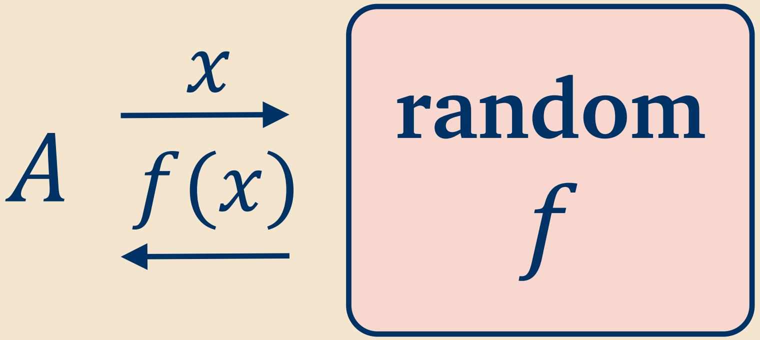

$A \xrightarrow{x}$

efficient simulation

If x is new:

- sample random y
- store (x, y)

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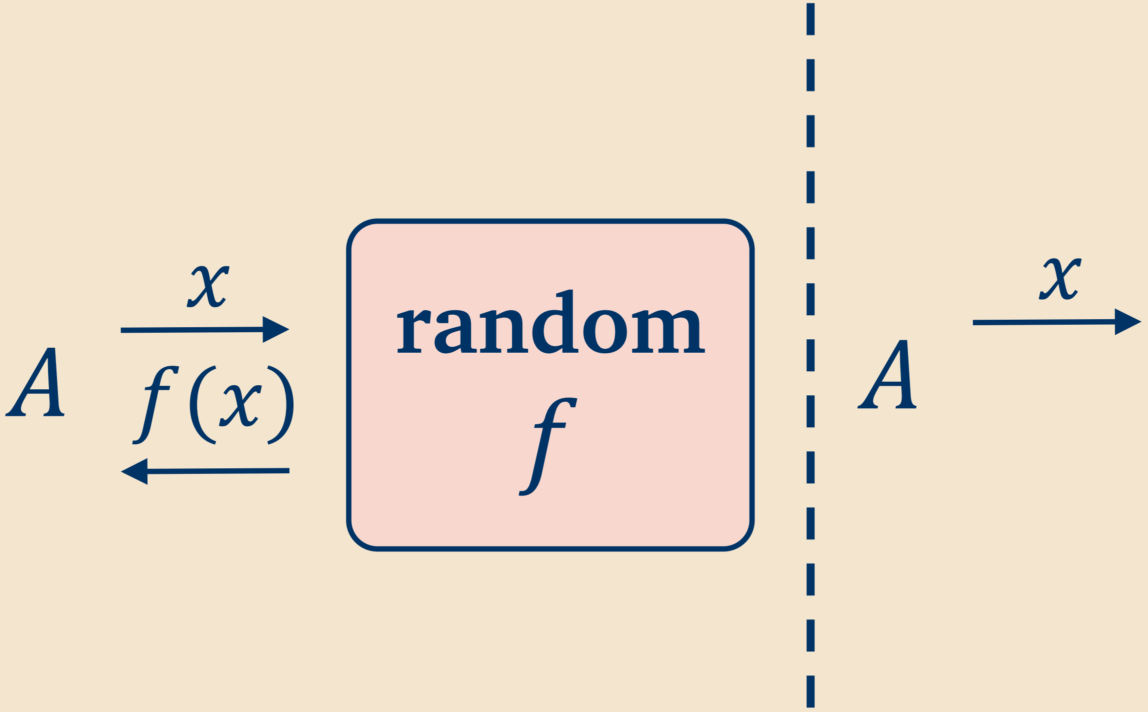

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$A \underset{f(x)}{\overset{x}{\rightleftarrows}}$ **random** f

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Clearly identical from $A's$ point of view!

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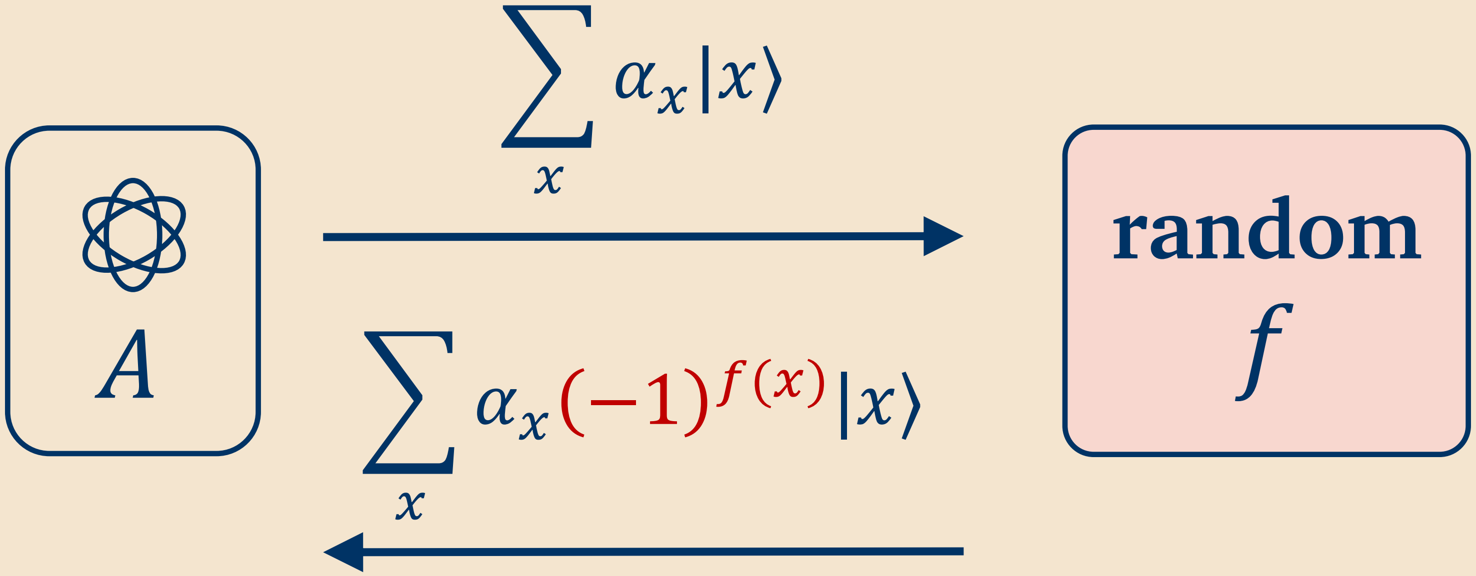

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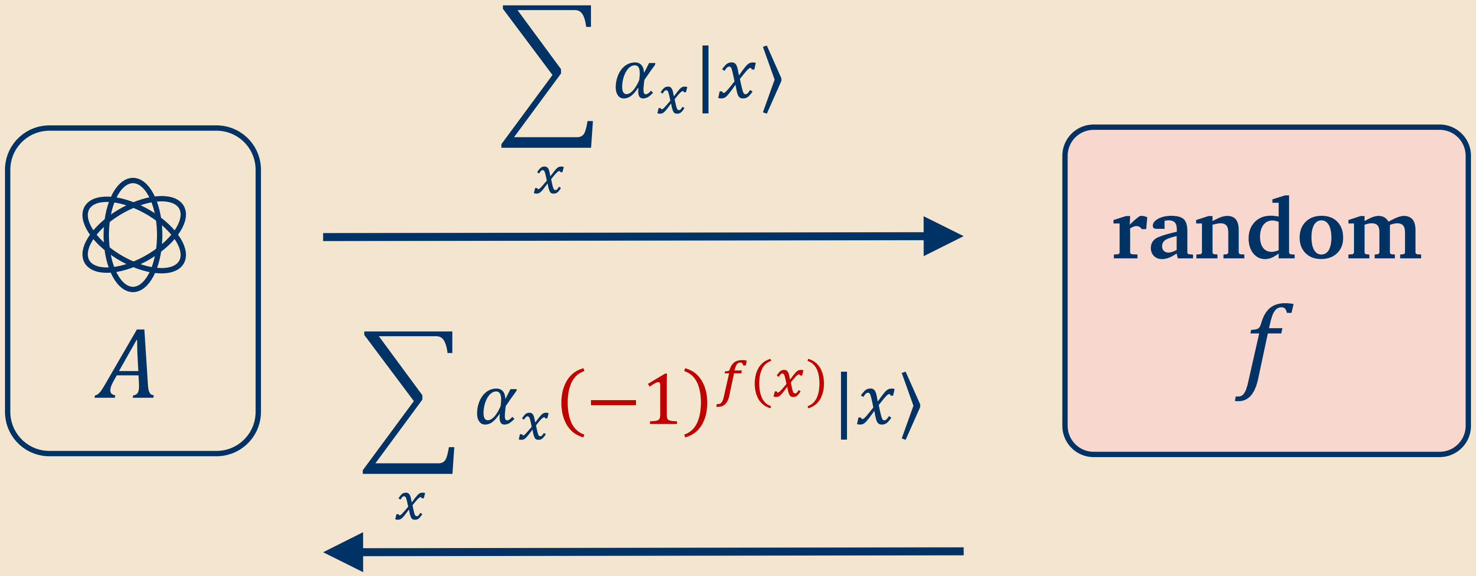

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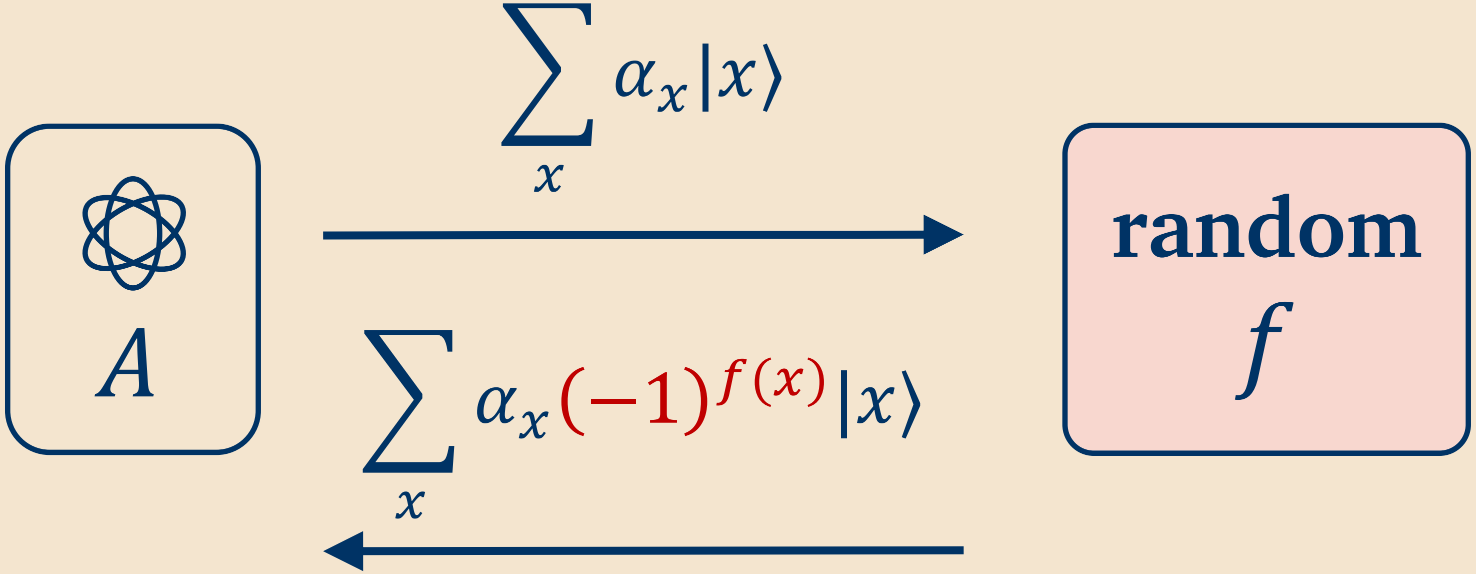

[Z19] simulates this via the **compressed oracle.**

Our **path-recording oracle** is a unitary analog of [Z19].

The path-recording oracle prO

$$|R\rangle\,|x\rangle \;\xrightarrow{\mathrm{prO}}\; \sum_{y \notin R} |y\rangle\,|R \cup \{(x, y)\}\rangle$$

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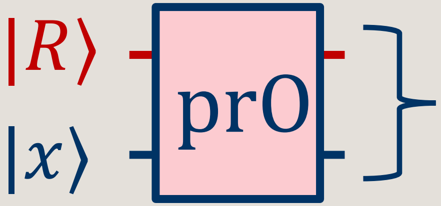

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- $R = \{(x_1, y_1), \dots, (x_t, y_t)\}$
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Note: prO is an isometry.

Intuition: $|y\rangle|R \cup \{(x,y)\}\rangle$ uniquely determines $|x\rangle|R\rangle$.

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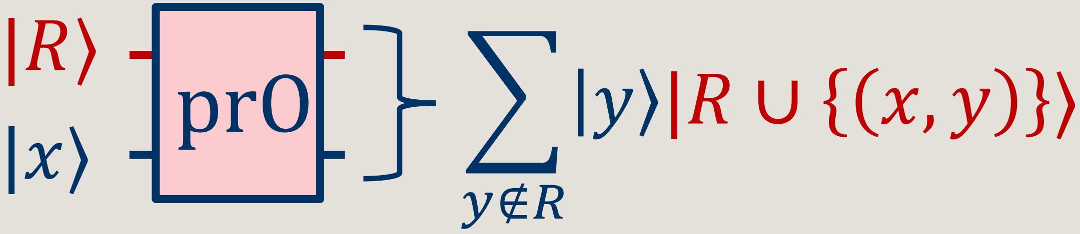

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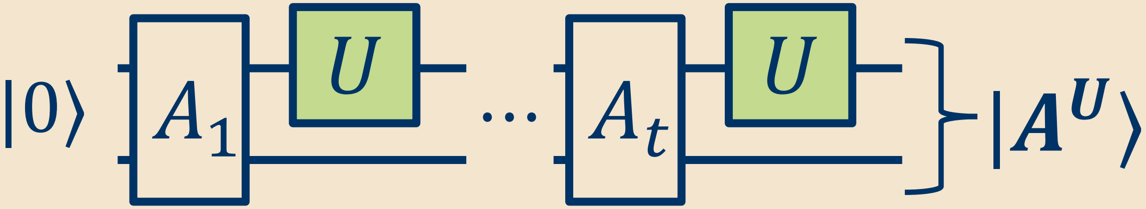
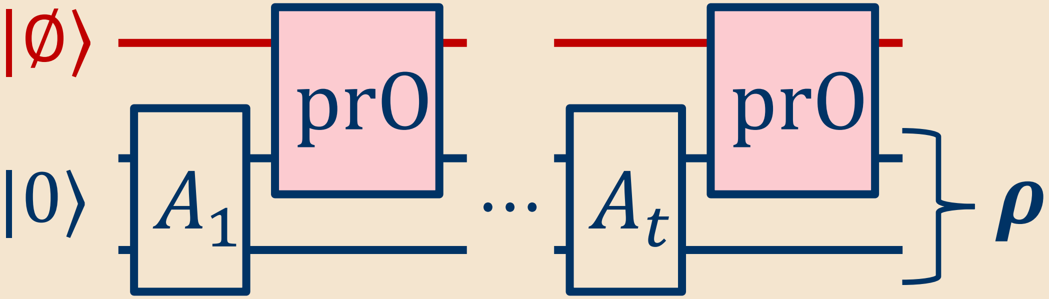
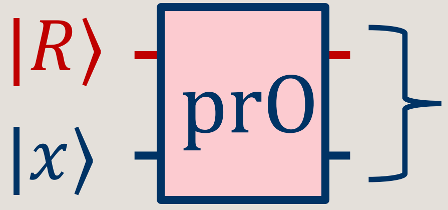


$U \leftarrow$ Haar



Efficient simulation

The path-recording oracle prO



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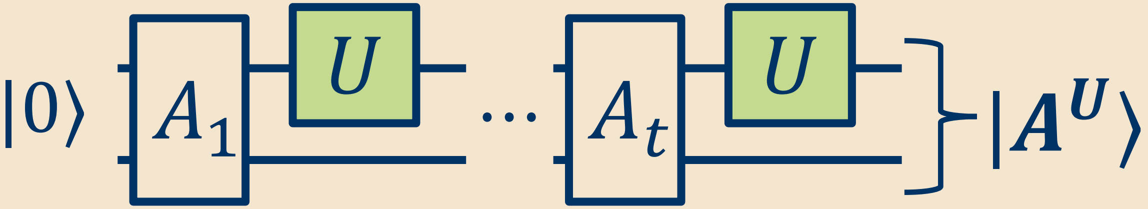

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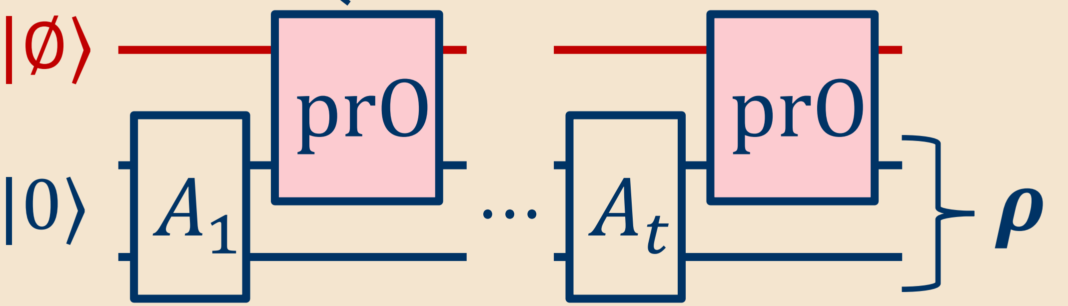

Efficient simulation

Claim: $\mathbb{E}_{U \leftarrow \mathbf{Haar}} |\boldsymbol{A}^U\rangle\langle \boldsymbol{A}^U|$ and $\boldsymbol{\rho}$ have trace distance $\leq t^2/2^n$.

Up next: a few examples

Example 1: one query on $|0\rangle$

$$|0\rangle \; - \boxed{U} - \; U|0\rangle$$

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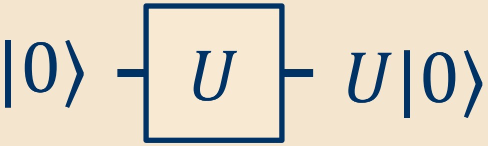

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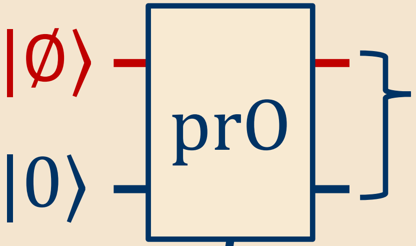

$\sum_y |y\rangle \otimes |\{(0,y)\}\rangle$

$$\mathrm{prO}\ |x\rangle|R\rangle = \sum_{y \notin R_Y} |y\rangle|R \cup \{(x,y)\}\rangle$$

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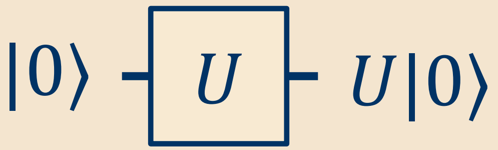

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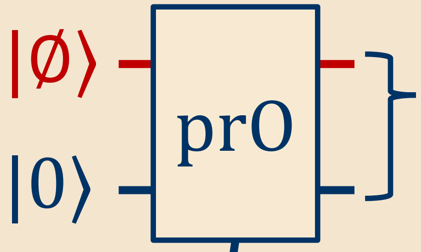

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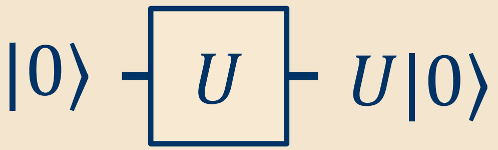

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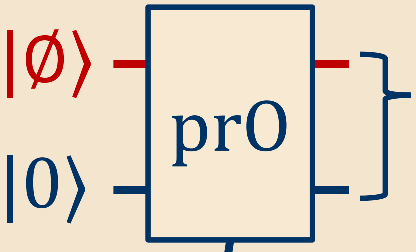
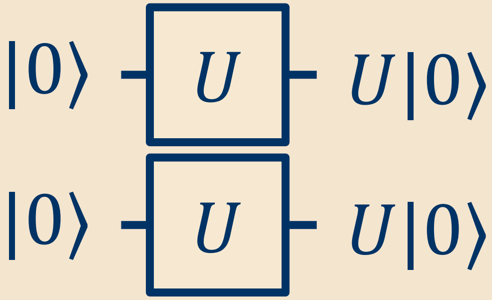


$$\sum_y |y\rangle \otimes \underbrace{|\{(0,y)\}\rangle}_{\text{trace out/measure}}$$

After tracing out: uniform mixture over $|y\rangle$, which is **maximally mixed**.

$$\boxed{\mathrm{prO}\,|x\rangle|R\rangle = \sum_{y \notin R_Y} |y\rangle|R \cup \{(x,y)\}\rangle}$$

Example 2: two queries on $|0\rangle$



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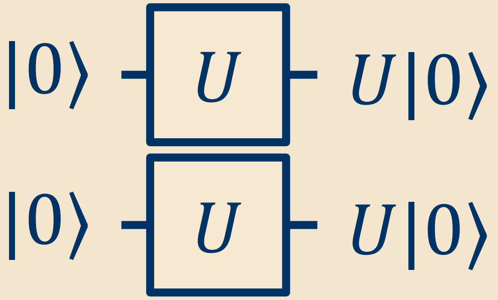

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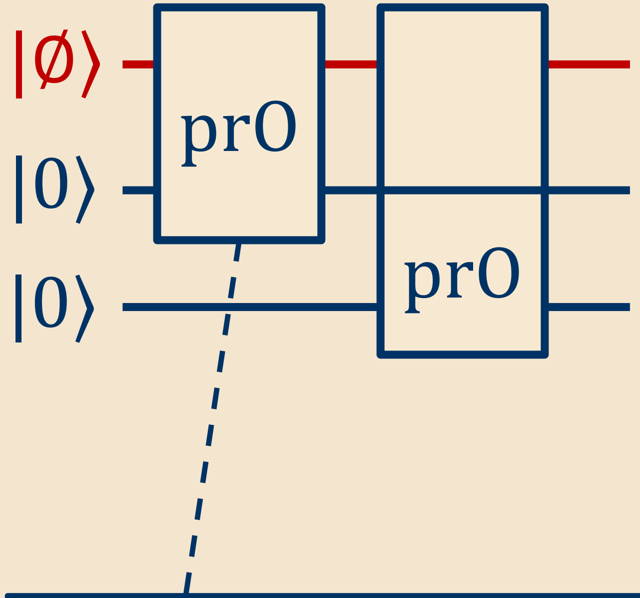

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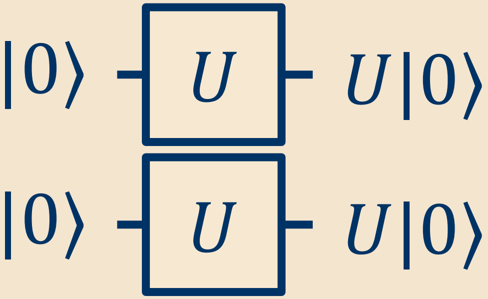

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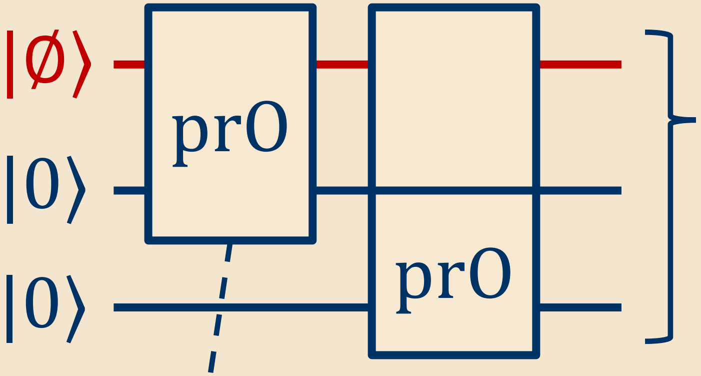

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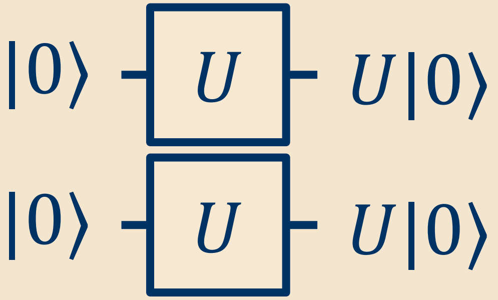

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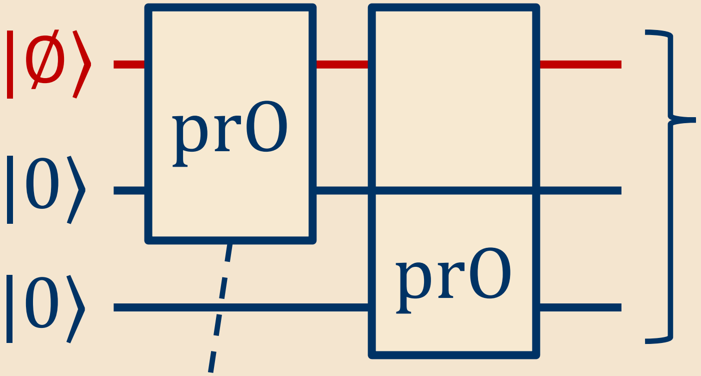

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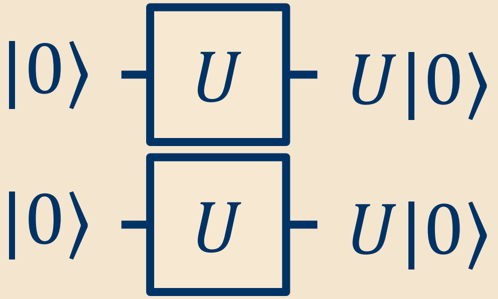

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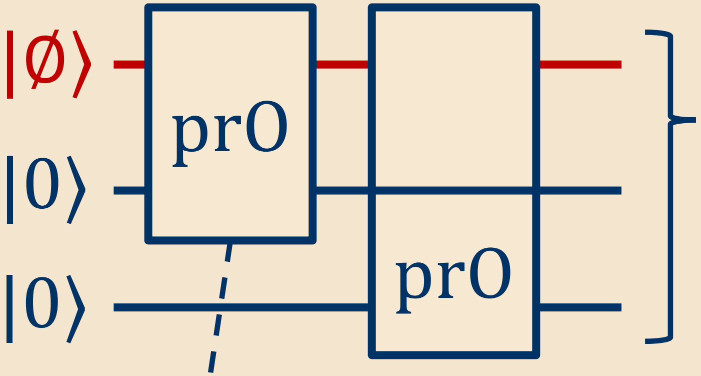

$$\sum_{y_1 \neq y_2} |y_1, y_2\rangle \otimes \underbrace{|\{(0, y_1), (0, y_2)\}\rangle}_{\text{trace out/measure}}$$

Remaining state: $|y_1, y_2\rangle + |y_2, y_1\rangle$ for random distinct y_1, y_2. This is **swap-invariant + almost maximally random**.

$$\text{prO}\,|x\rangle|R\rangle = \sum_{y \notin R_Y} |y\rangle|R \cup \{(x, y)\}\rangle$$

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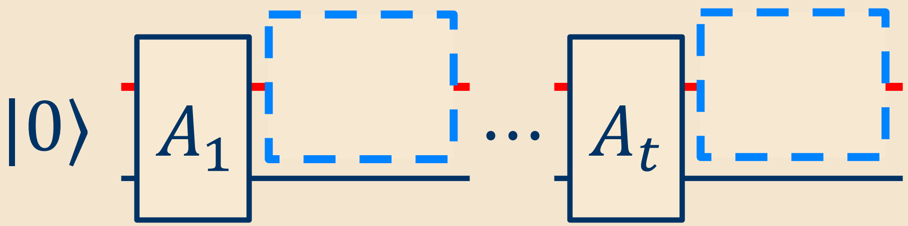

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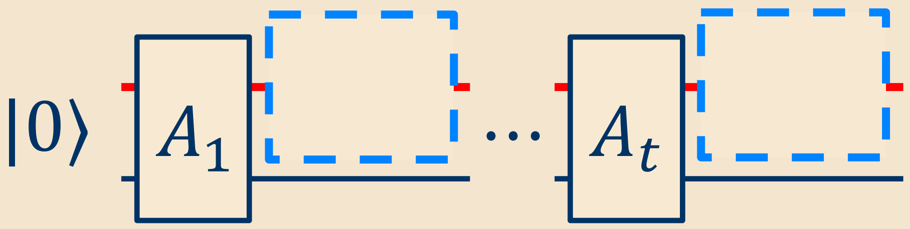

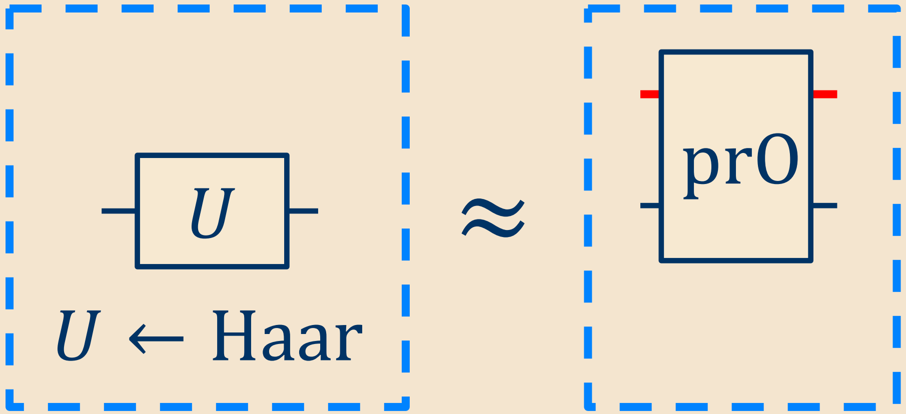

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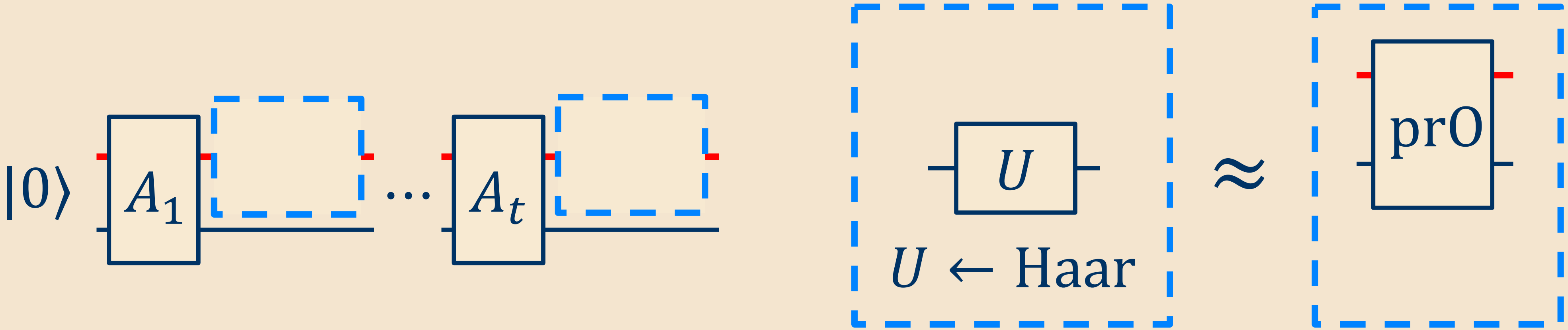

The plan: hybrid argument.

The same proof will also show existence of PRUs!

$U \leftarrow \text{Haar}$

Hybrid 0

$$-\boxed{U}-$$

$$U \leftarrow \text{Haar}$$

$$-\boxed{PF}-\boxed{U}-$$

$$U \leftarrow \text{Haar}$$
$$P \leftarrow S_N$$
$$F \leftarrow \{\pm 1\}^N$$

$$\text{Hybrid } 0 \quad \equiv \quad \text{Hybrid } 1$$

$$-\boxed{U}-$$

$U \leftarrow \mathrm{Haar}$

$$-\boxed{PF}-\boxed{U}-$$

$U \leftarrow \mathrm{Haar}$
$P \leftarrow S_N$
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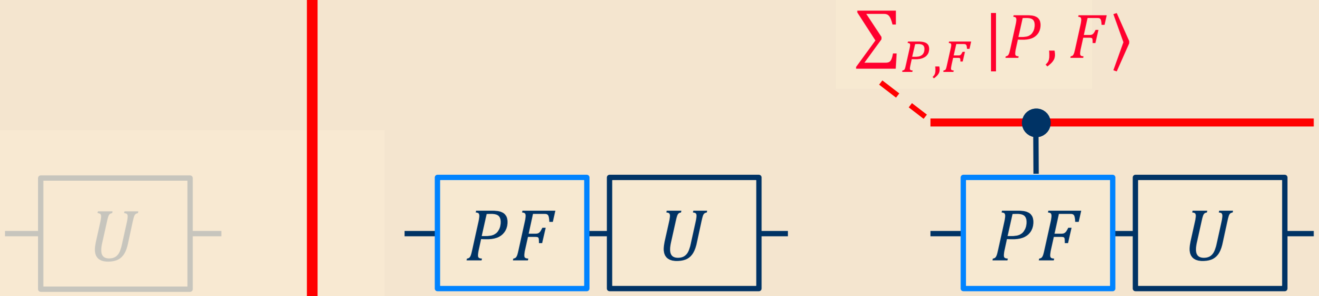
Hybrid 0 $\equiv$ Hybrid 1

Step 1: insert random permutation P random ± 1 diagonal F.

$$P = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \qquad F = \begin{pmatrix} +1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$



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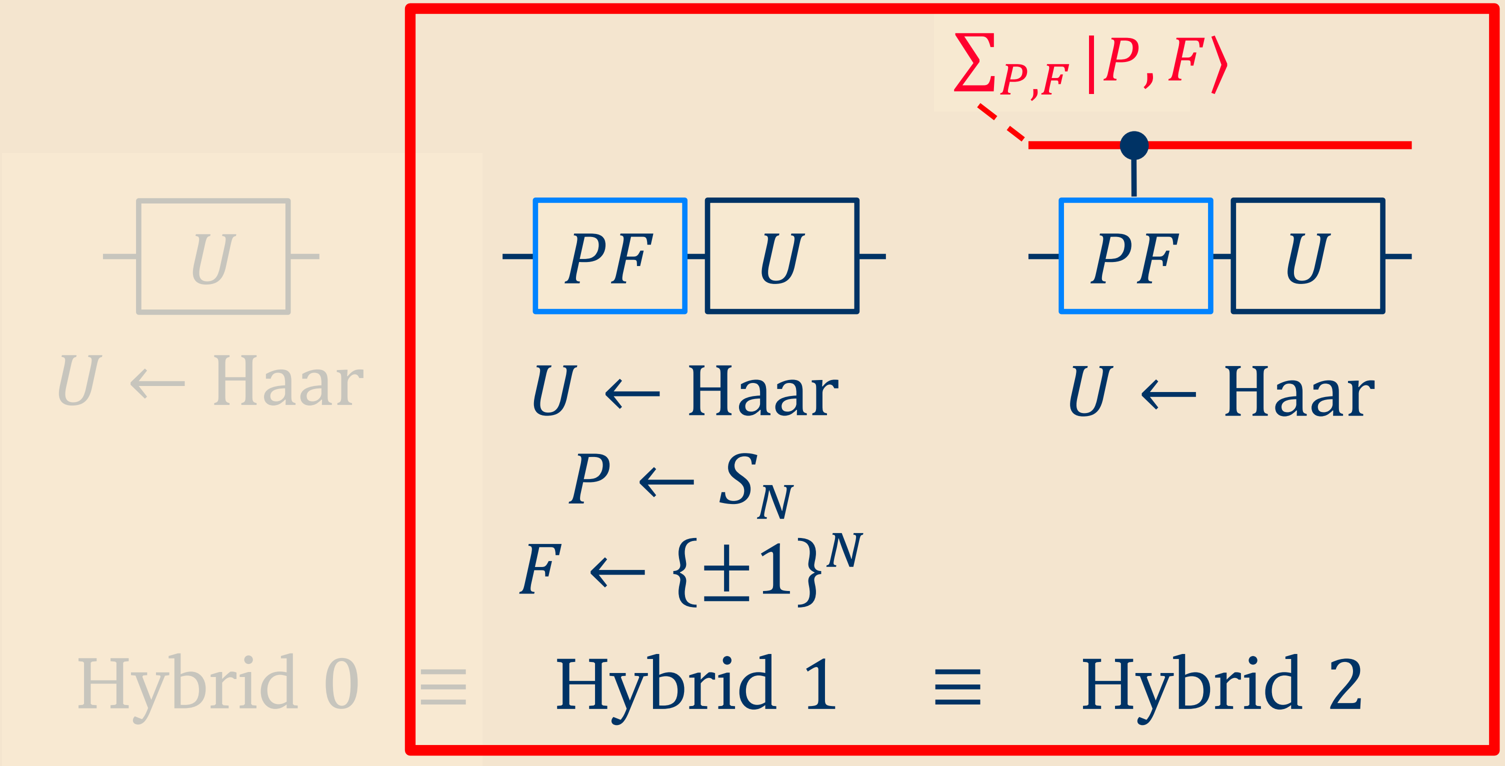
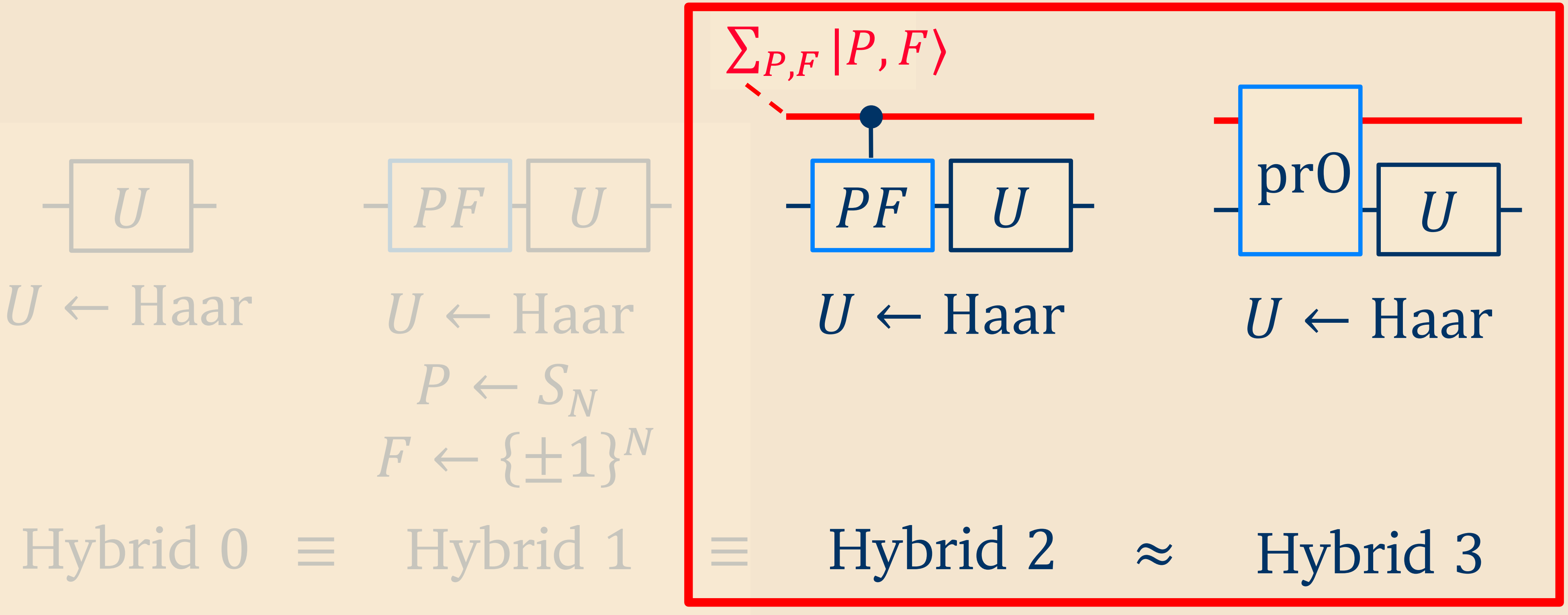
Hybrid 0 $\equiv$ Hybrid 1 $\equiv$ Hybrid 2



Step 2: replace random P, F with a purification.

- Initialize external/ancilla system to $\sum_{P,F} |P, F\rangle$
- On each query, apply $P \cdot F$ **controlled** on $|P, F\rangle$



$\sum_{P,F} |P, F\rangle$
U
PF
U
PF
U
prO
U
$U \leftarrow \text{Haar}$
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$U \leftarrow \text{Haar}$
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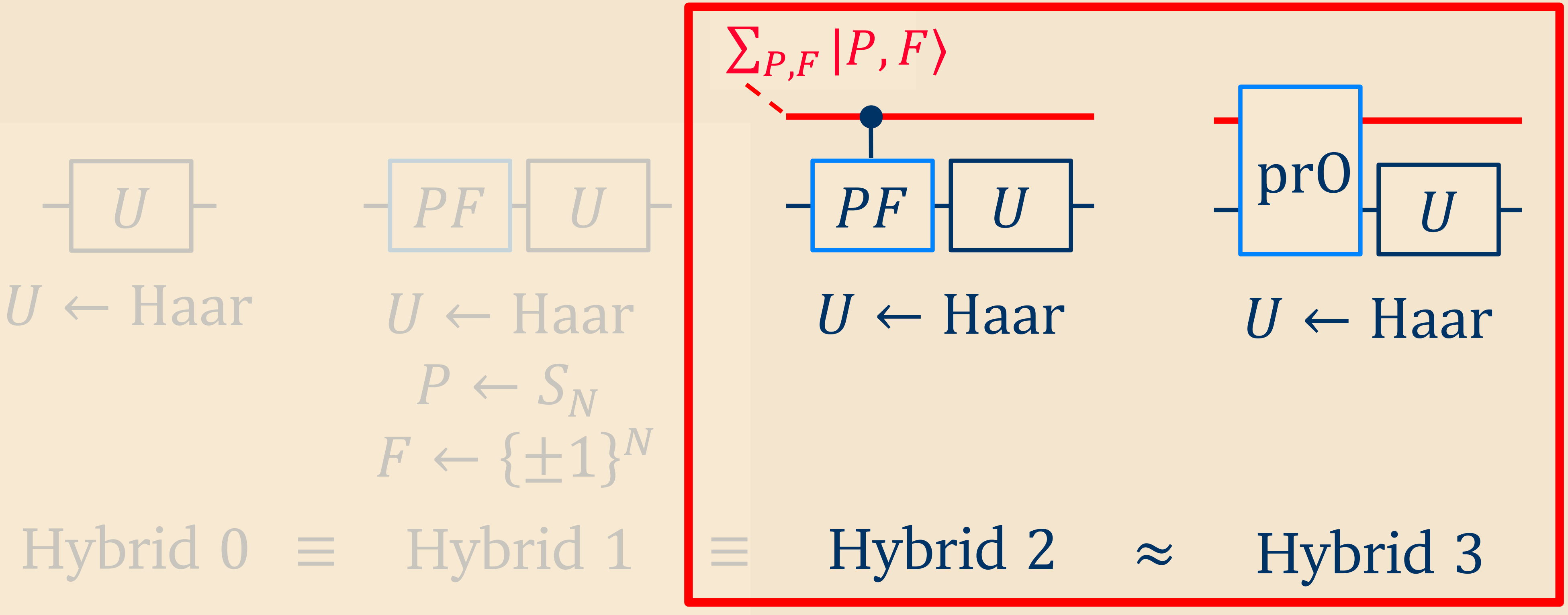
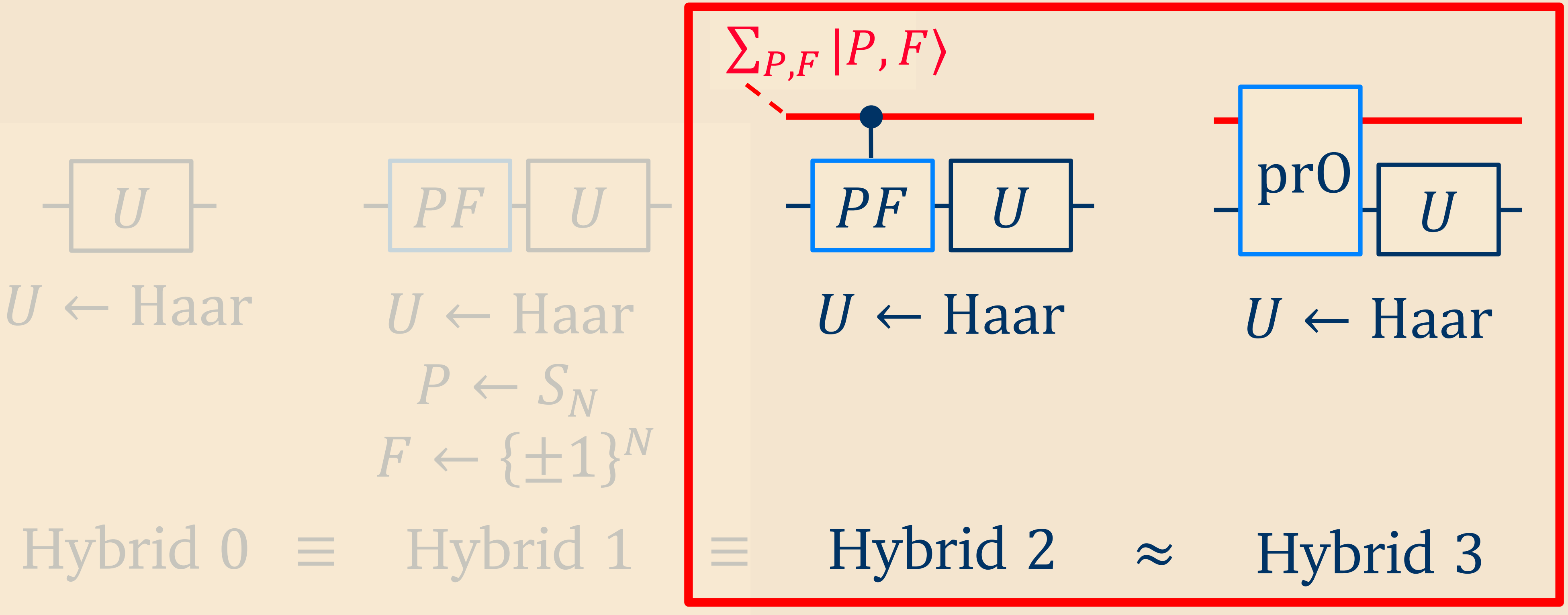
Hybrid 0 $\equiv$ Hybrid 1 $\equiv$ Hybrid 2 $\approx$ Hybrid 3



Step 3: For any $R = \{(x_1, y_1), \ldots, (x_t, y_t)\}$ can define $|\Phi_R\rangle$ s.t.

$$\text{ctl-PF} \cdot |x\rangle|\Phi_R\rangle = \sum_{y \notin R_Y} |y\rangle|\Phi_{R \cup \{(x,y)\}}\rangle$$



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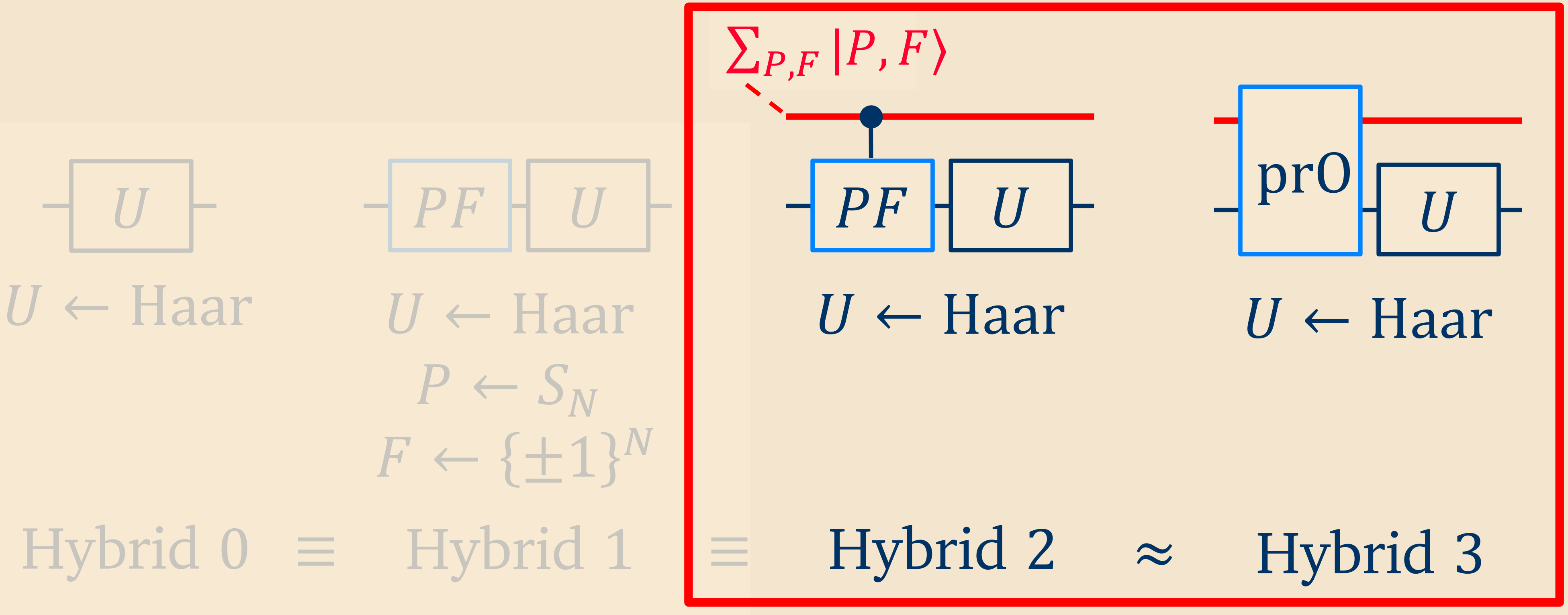

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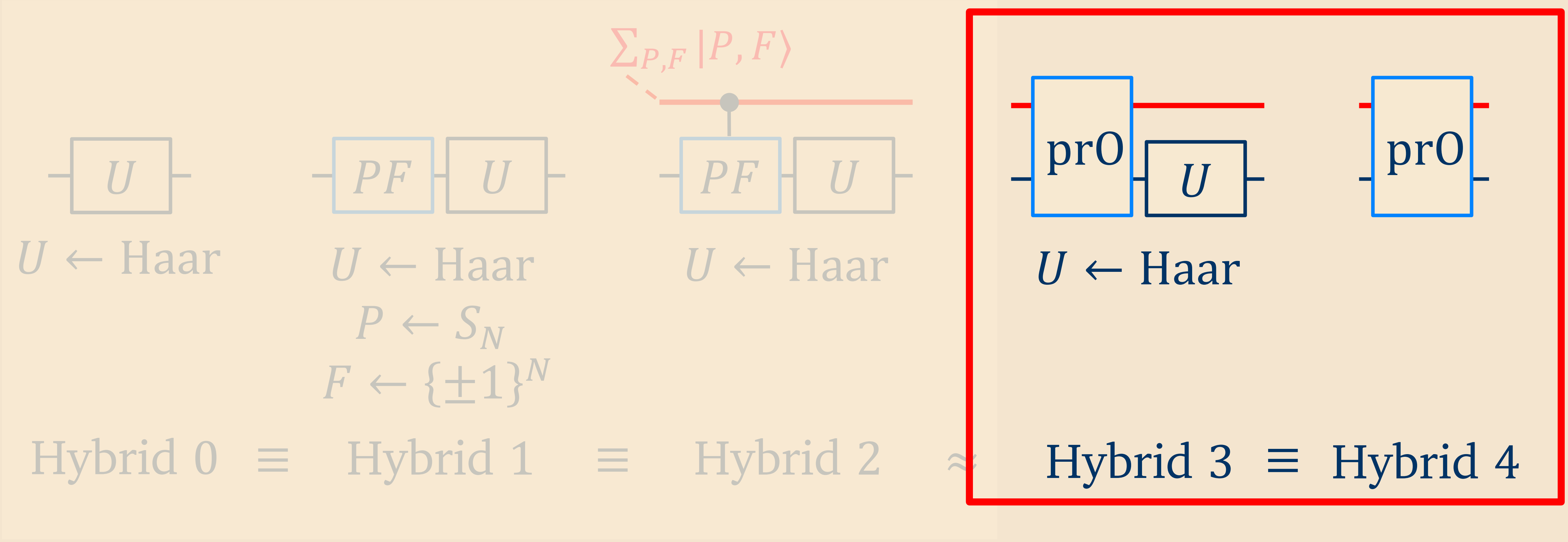
- **Intuition:** ctl-PF behaves like prO, up to relabeling $|\Phi_R\rangle \mapsto |R\rangle$
- Actually, $\{|\Phi_R\rangle\}_R$ aren't fully orthogonal. But composing with $U \leftarrow$ (2-design) makes the "non-orthogonal" ones hard to find.



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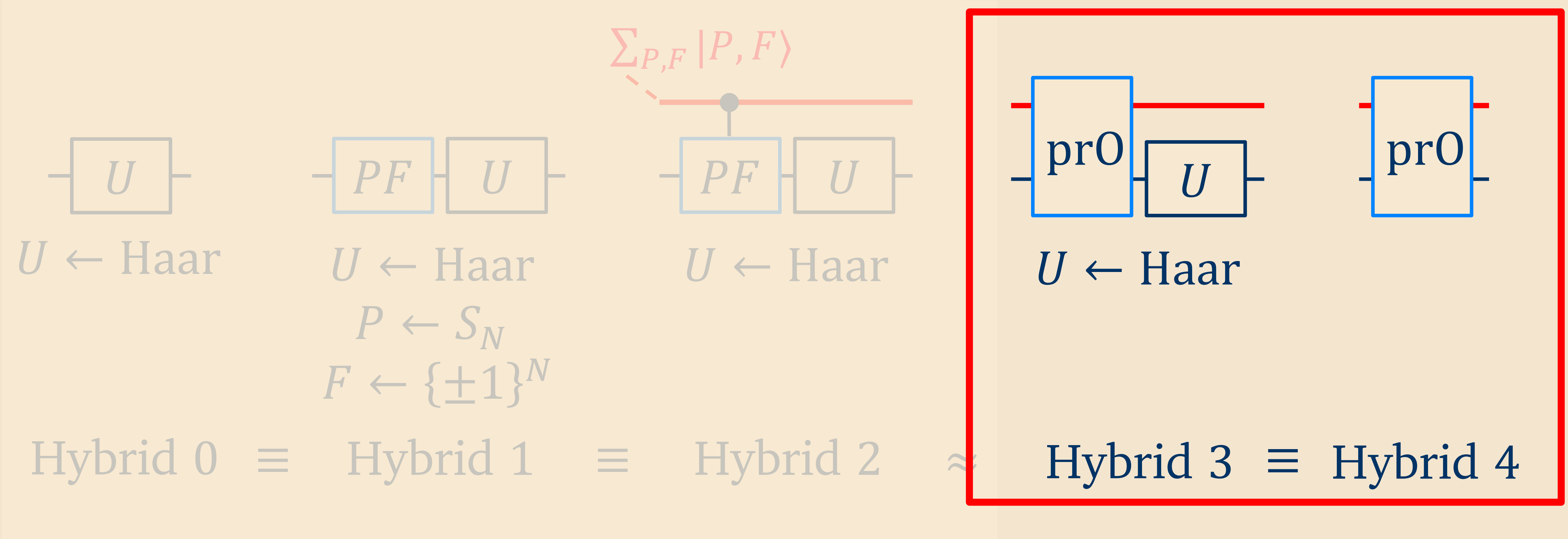
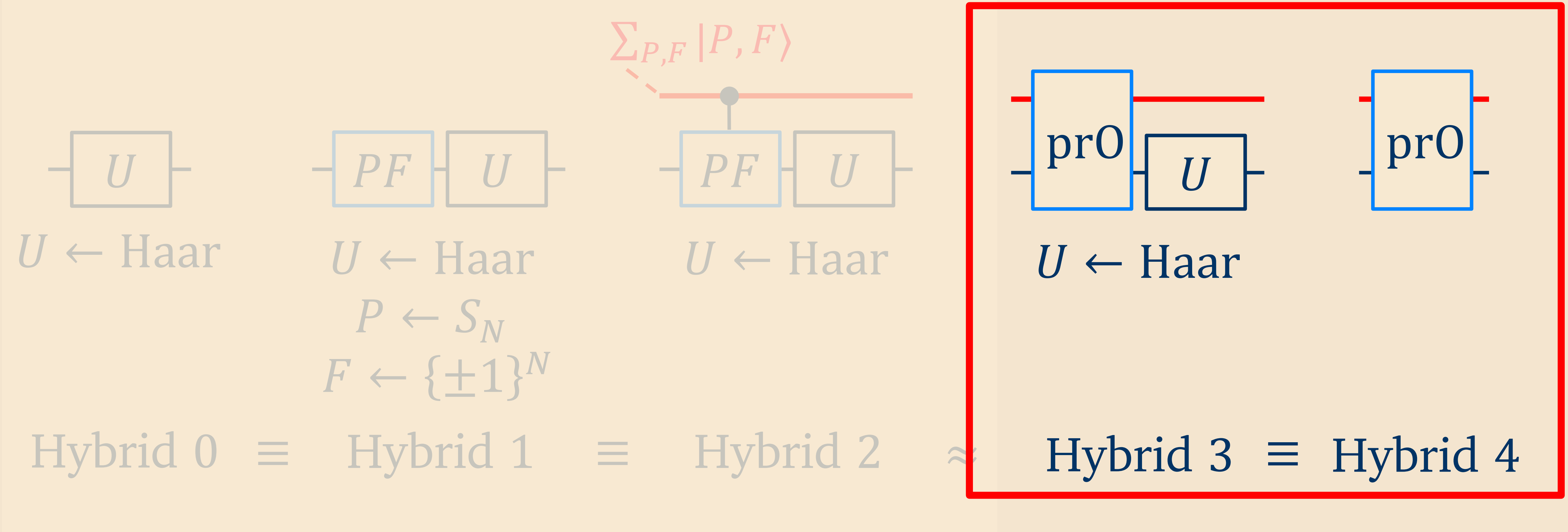


Step 4: Turns out prO has the following magical property:



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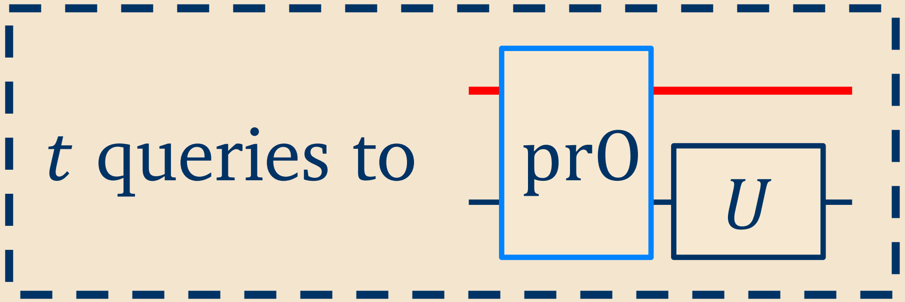

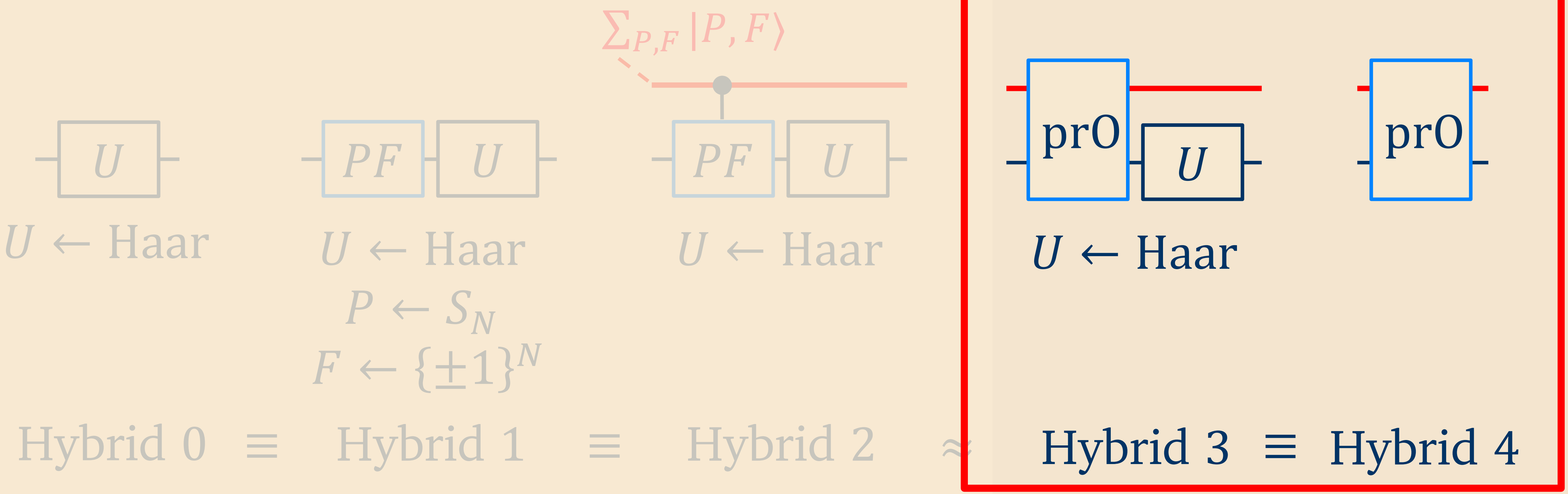

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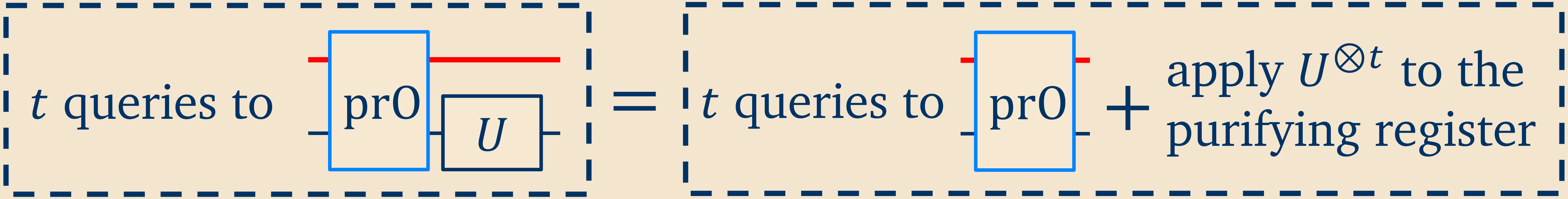

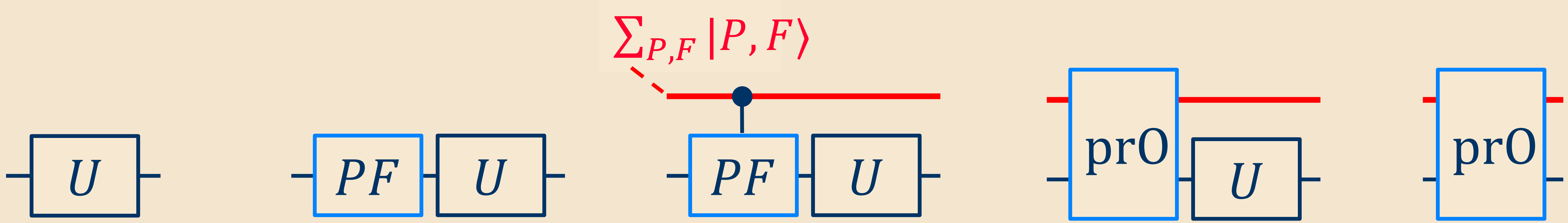

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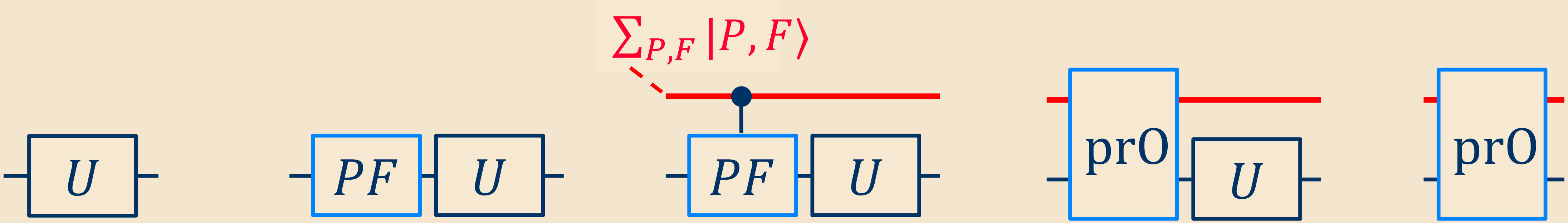

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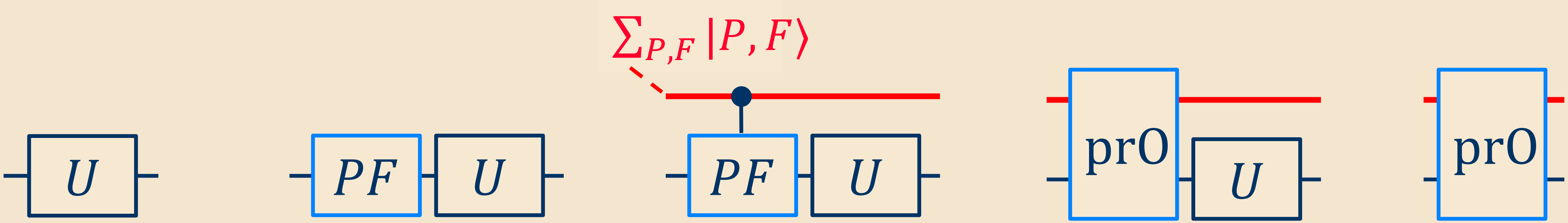

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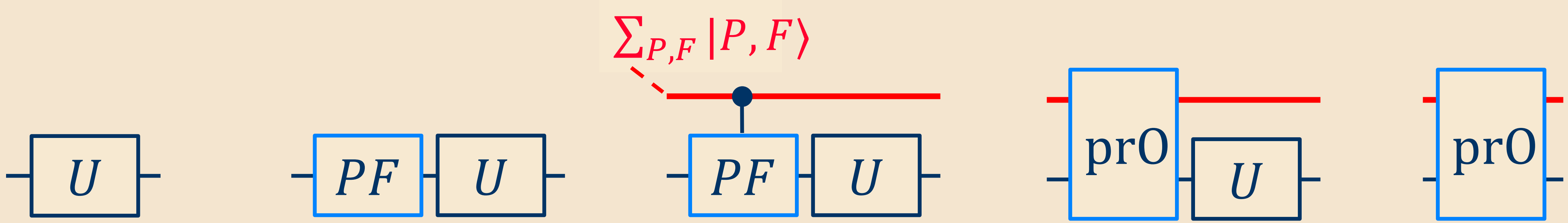

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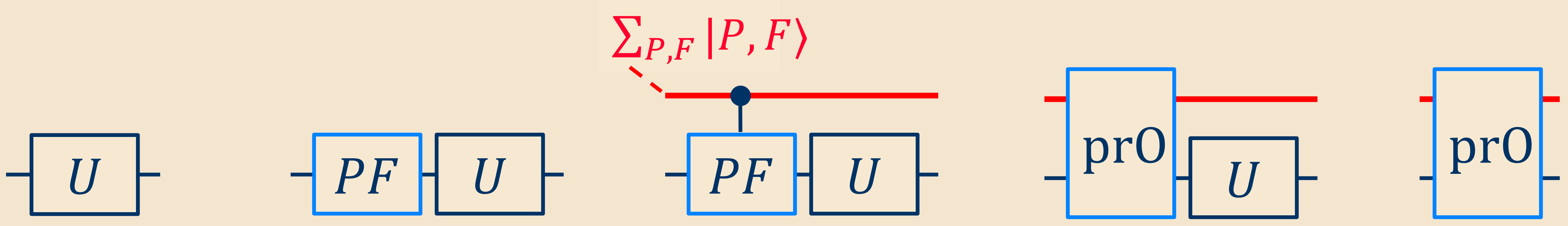

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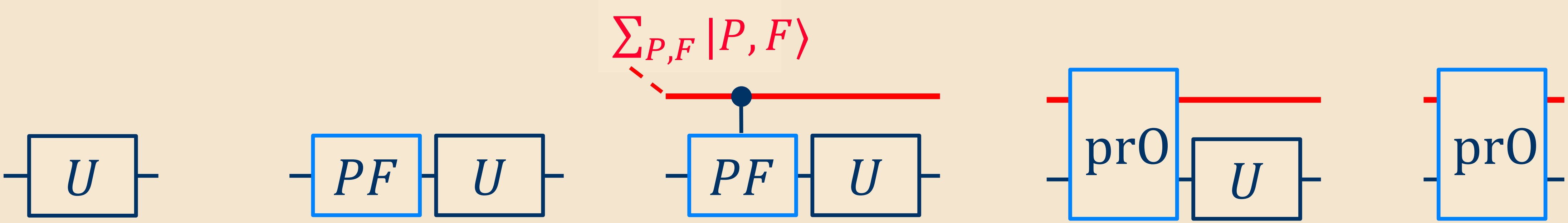

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Finally, replace P and F with pseudorandom.

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Already several applications:

- [MH24]: elementary proof of [SHH24] gluing lemma
- [ABGL24]: compress PRU key length + other results

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Thanks!