How to Construct Random Unitaries

Fermi Ma

joint work with Hsin-Yuan Huang

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Philosophy: *"When good choices abound, guessing randomly can be surprisingly fruitful" (Quanta Magazine)*

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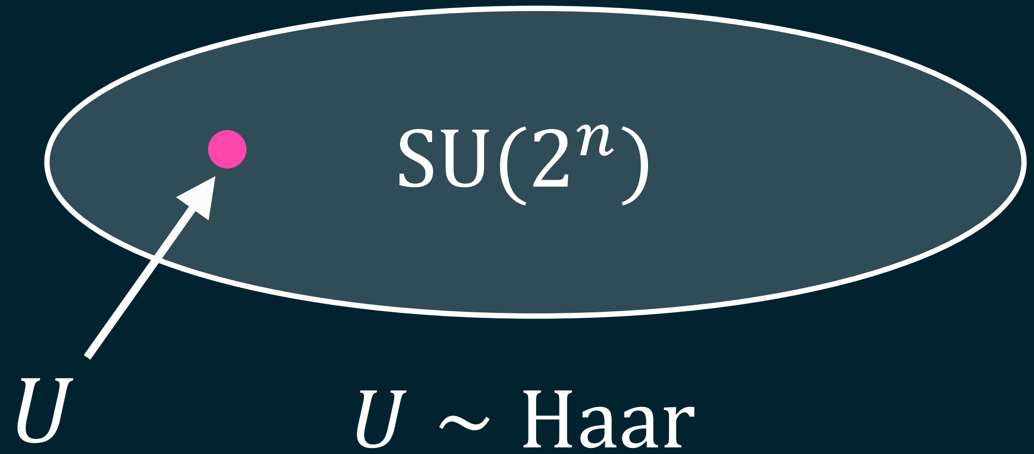

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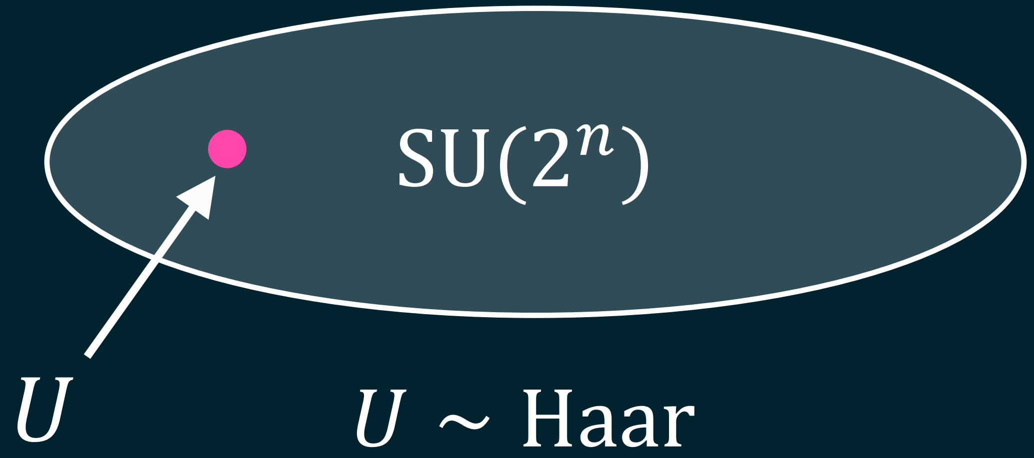

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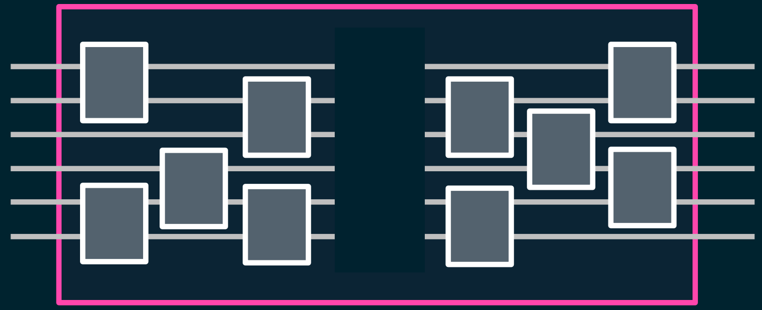

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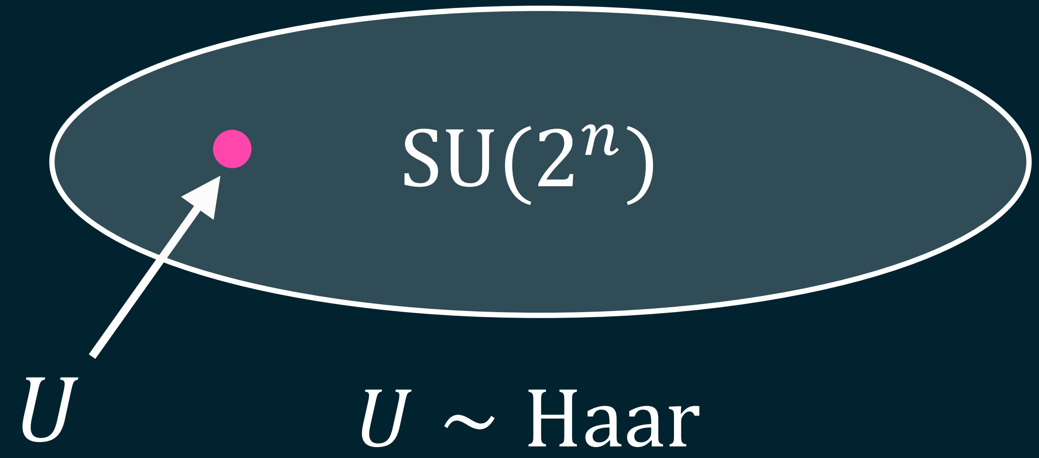

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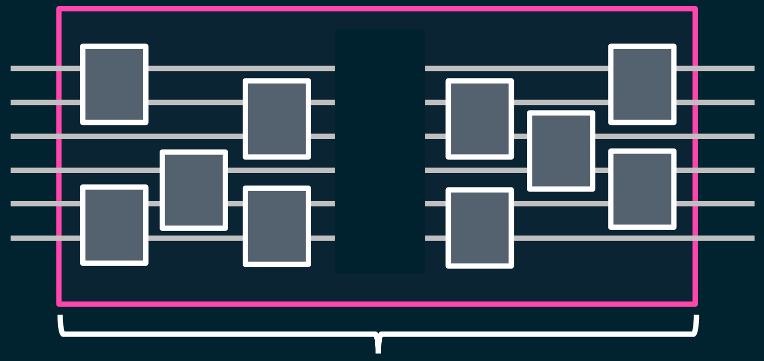

$\exp(n)$ depth

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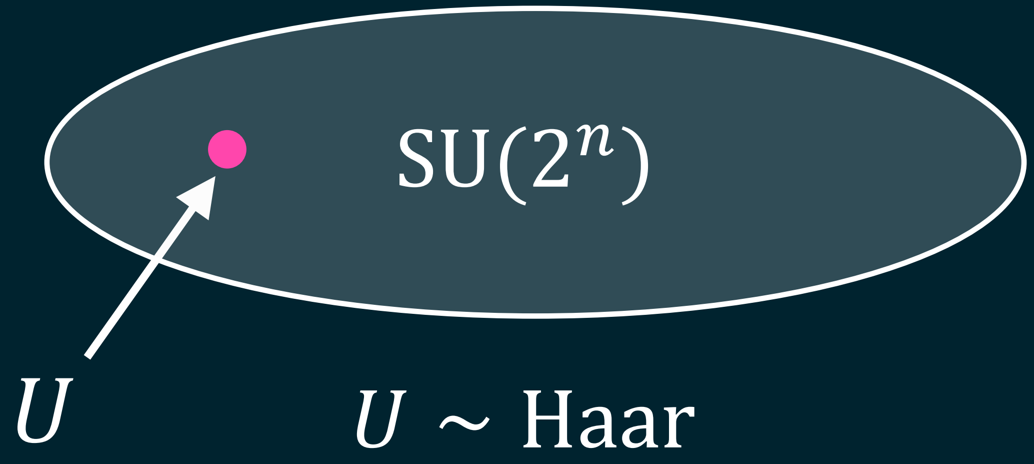

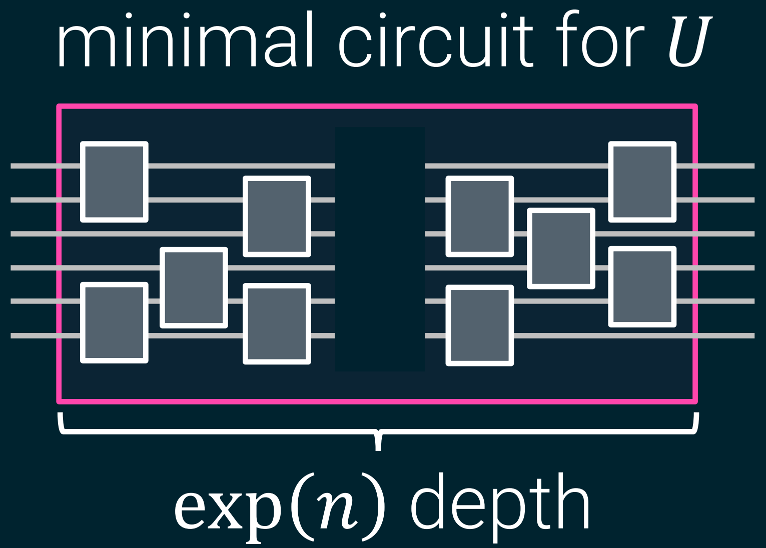

So even if a Haar-random unitary "solves" your problem, this often isn't good enough!

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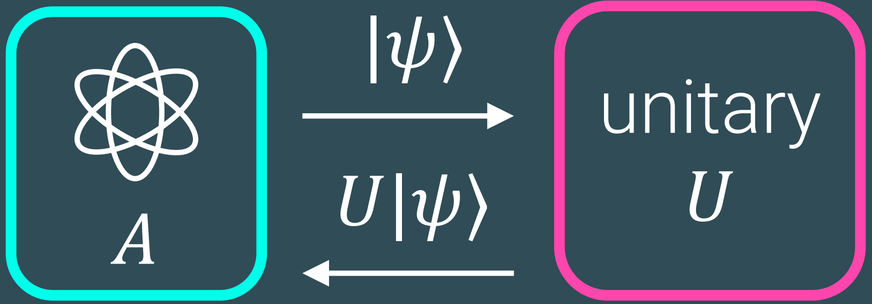

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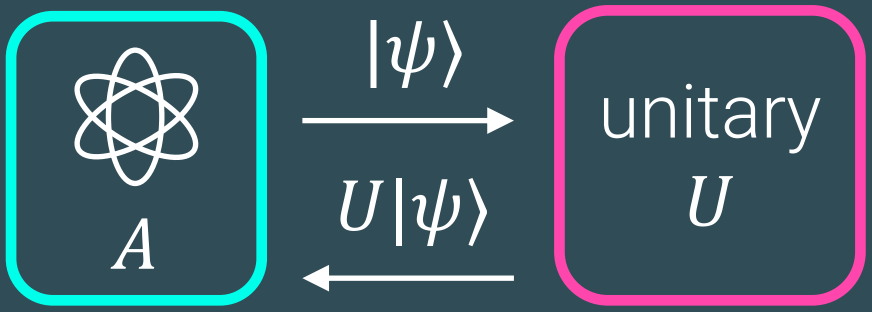

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Classical analogue: pseudorandom functions (PRFs) or pseudorandom permutations (PRPs)

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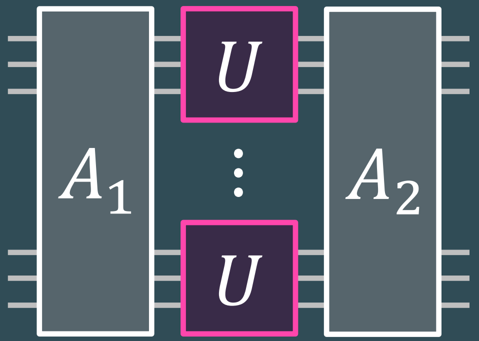

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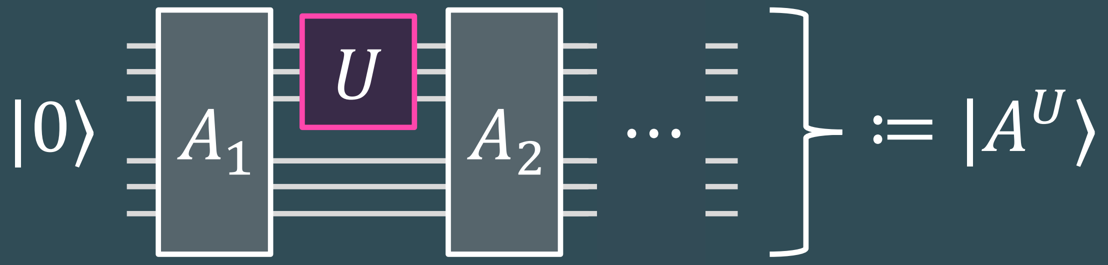

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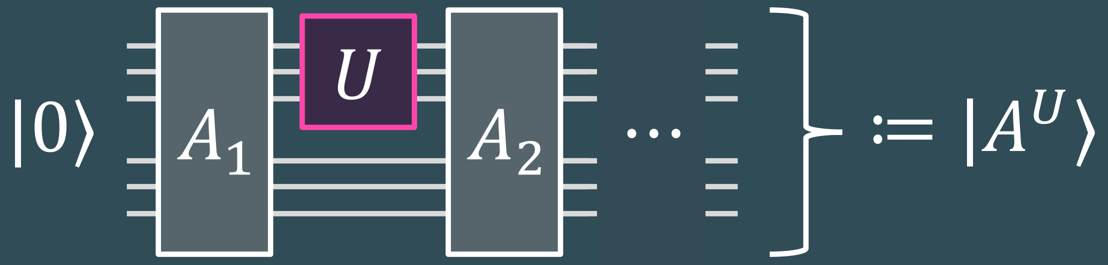

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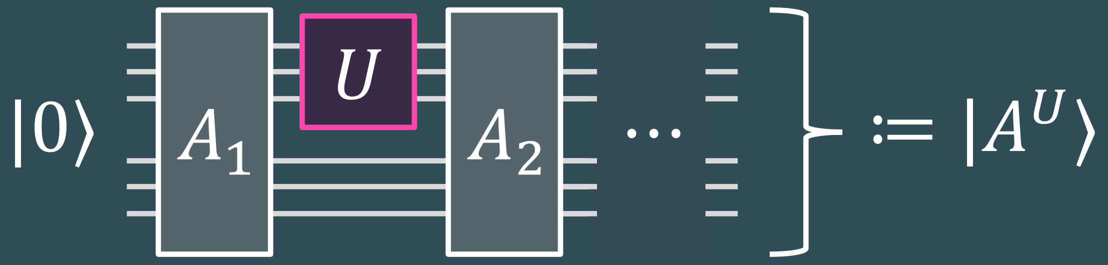

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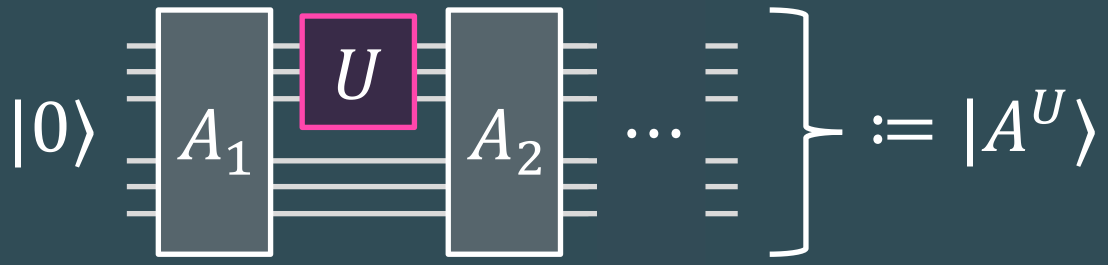

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e.g., Weingarten calculus

Theorem 3.1. *Let* k *be a positive integer. For any permutation* $\sigma \in \mathcal{S}_k$ *and nonnegative integer* g, *we have*

$$(k-1)^g \#P(\sigma, |\sigma|) \leq \#P(\sigma, |\sigma| + 2g) \leq (6k^{7/2})^g \#P(\sigma, |\sigma|).$$

Theorem 3.2. *For any* $\sigma \in \mathcal{S}_k$ *and* $d > \sqrt{6}k^{7/4}$,

$$\frac{1}{1 - \frac{k-1}{d^2}} \leq \frac{(-1)^{|\sigma|} d^{k+|\sigma|} \mathrm{Wg}^U(\sigma, d)}{\#P(\sigma, |\sigma|)} \leq \frac{1}{1 - \frac{6k^{7/2}}{d^2}}.$$

In addition, the l.h.s inequality is valid for any $d \geq k$.

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Theorem: PRUs exist (assuming one-way functions)

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- analyze random unitaries using purification
- **we also show:** any algorithm that queries a Haar-random U can be efficiently implemented (to inverse-exp error)

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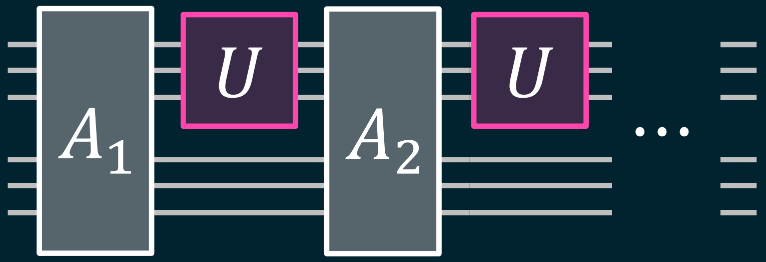

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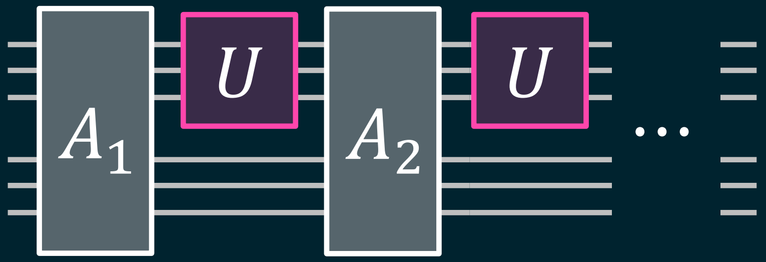

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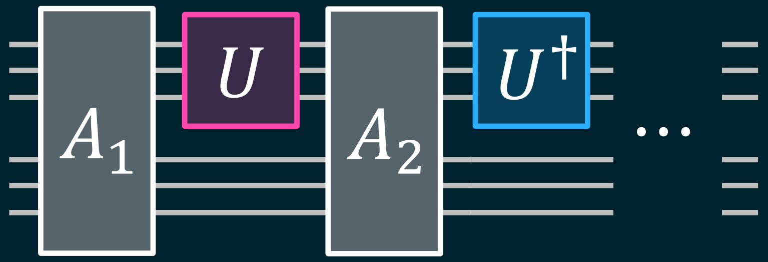

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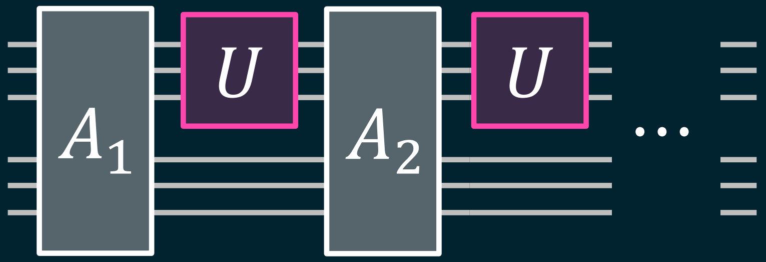

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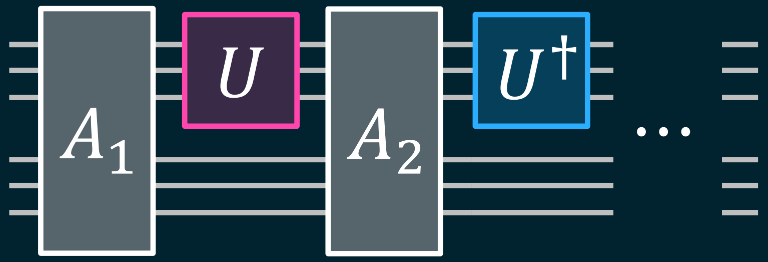

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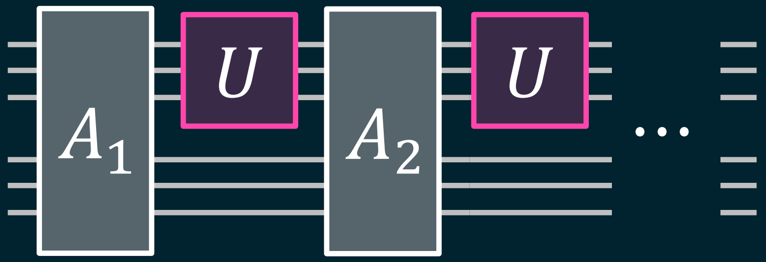

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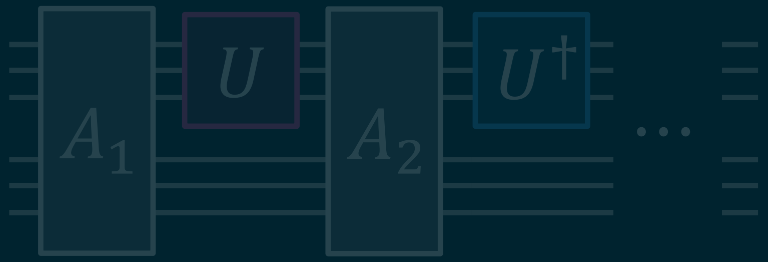

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(3) two proofs at once:
- our simulator works
- PRUs exist

Up next:

How to simulate a random **function**

Simulating a random **function**

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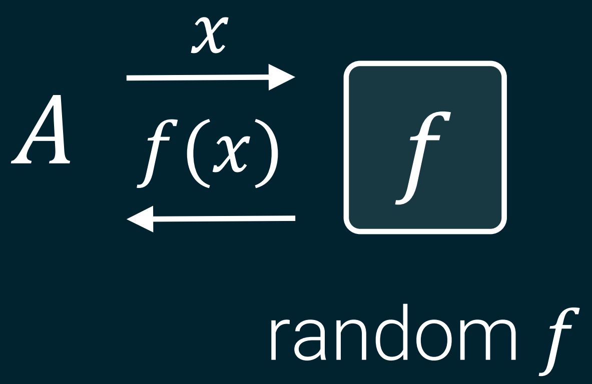

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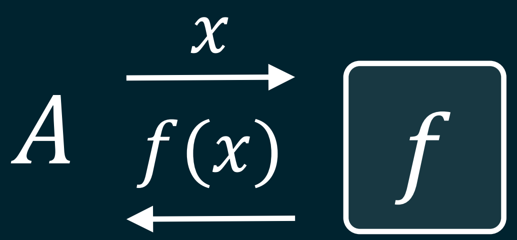

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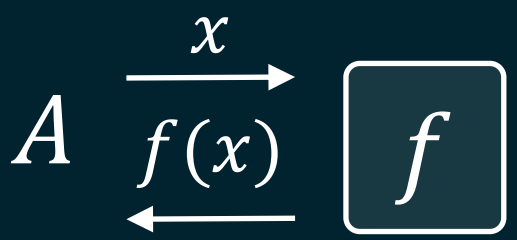

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$A \xrightarrow{x} \boxed{f}$, $\boxed{f} \xrightarrow{f(x)} A$

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Simulation

A

Initialization: $R = \emptyset$

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$A \underset{f(x)}{\overset{x}{\rightleftarrows}} \boxed{f}$

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$A \xrightarrow{x}$

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(polynomial-time + stateful)

Simulating a random function

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What if A^f is quantum and queries f in **superposition**?

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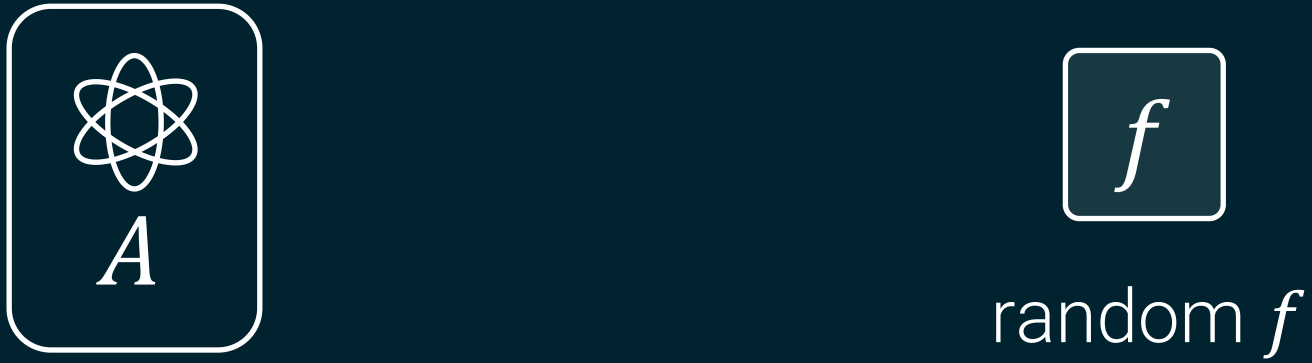

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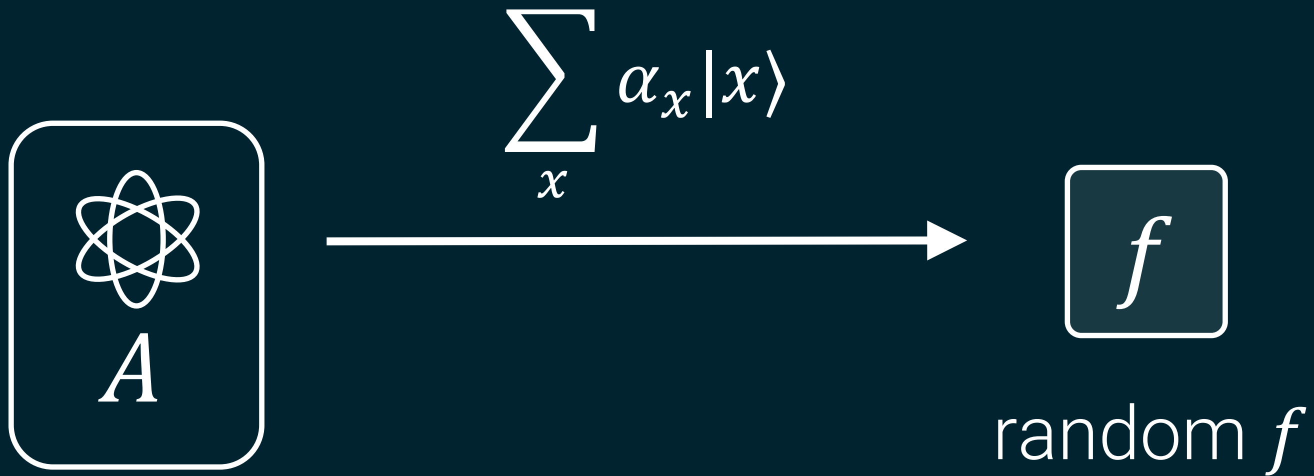

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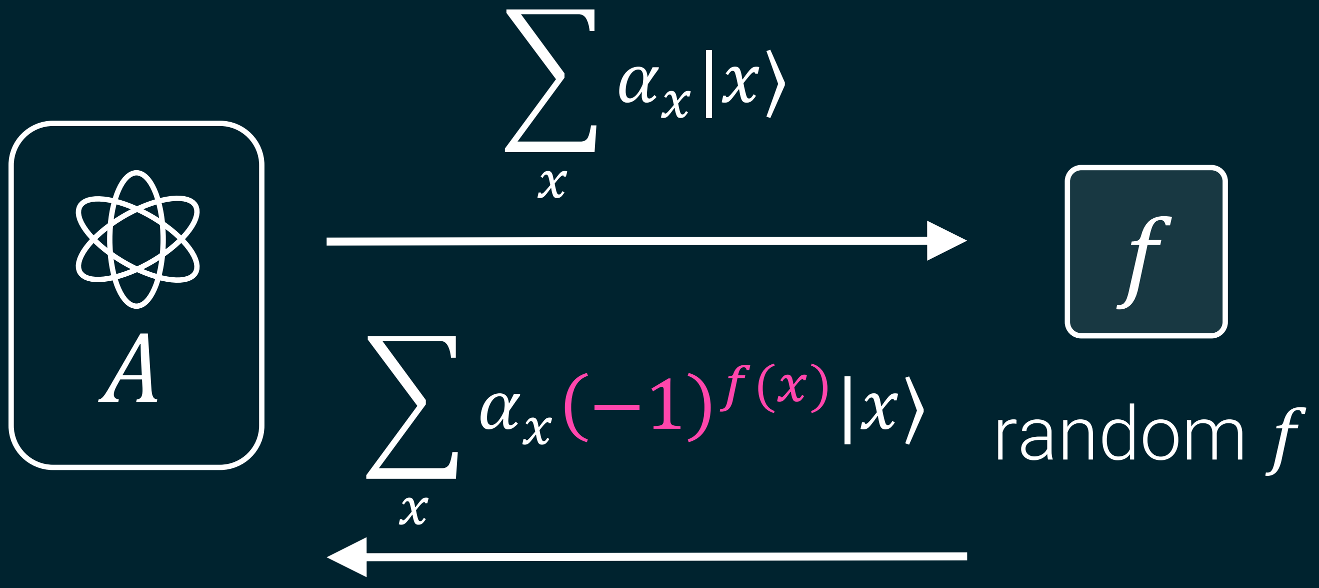

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$$\sum_x \alpha_x |x\rangle$$

A

f

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random f

Unclear how to sample $f(x)$ “on the fly.”

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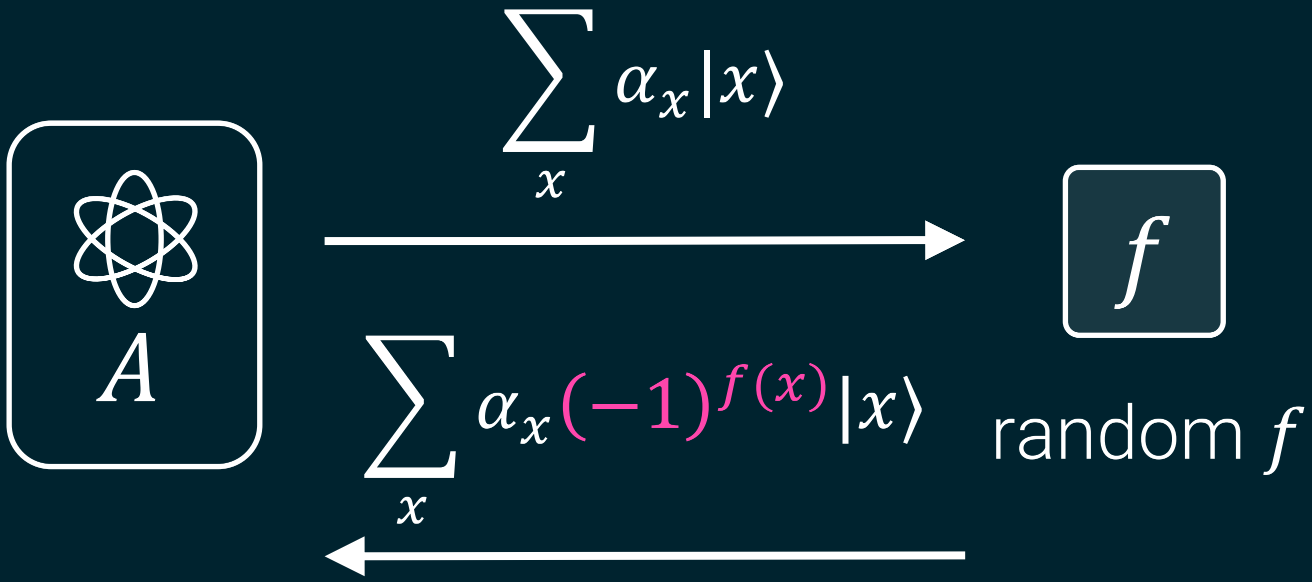

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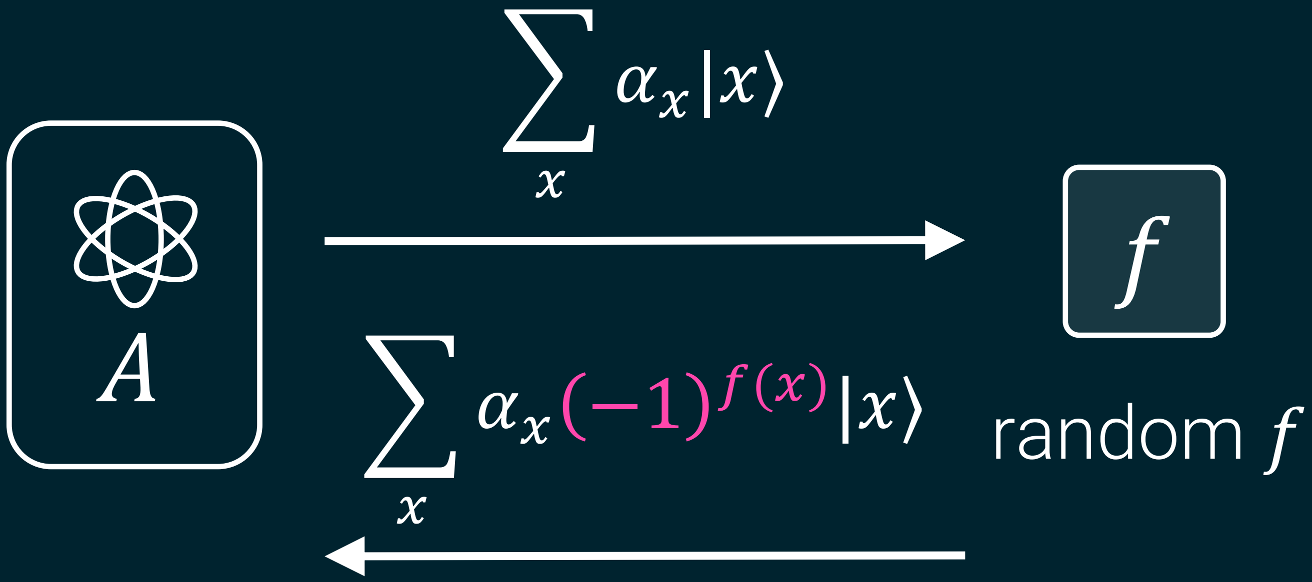

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Solution: the compressed oracle [Zhandry18]

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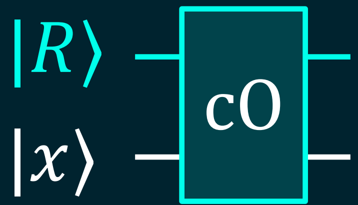

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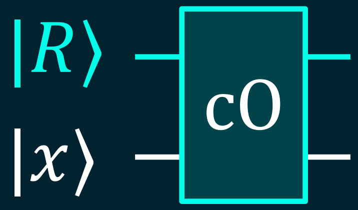

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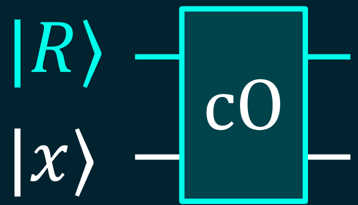

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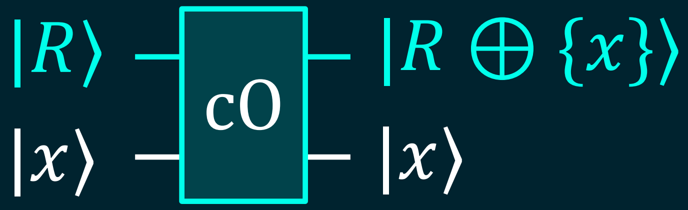

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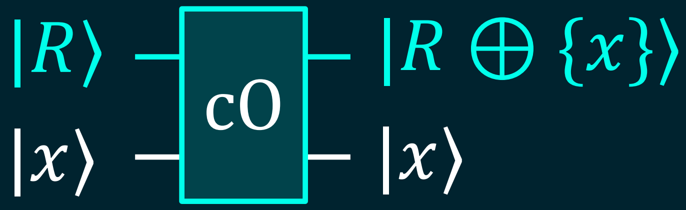

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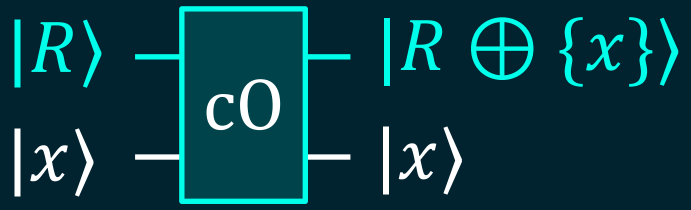

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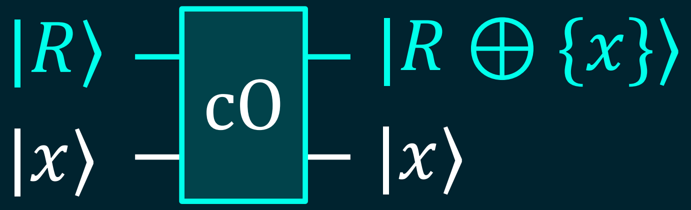

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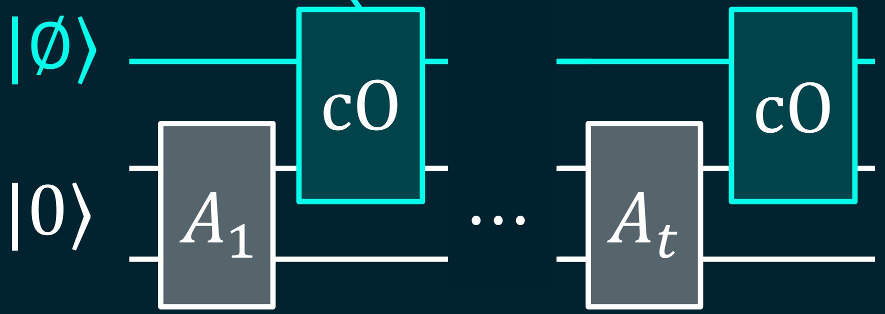

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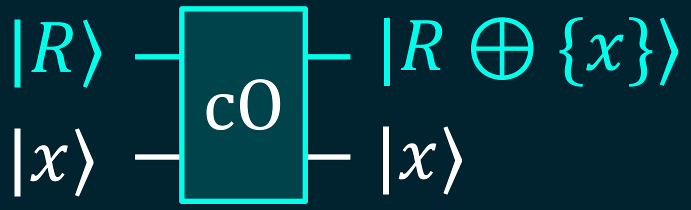

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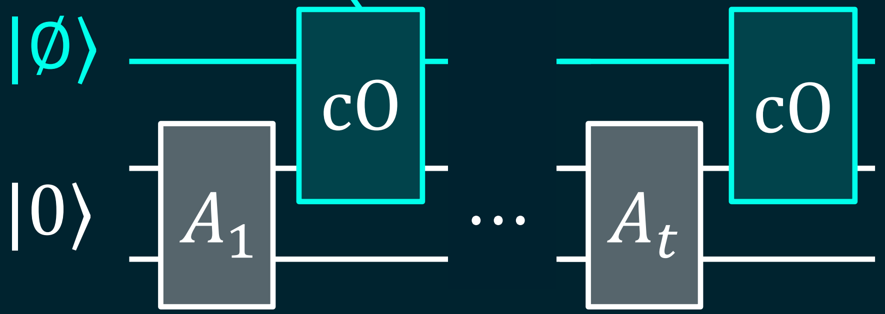

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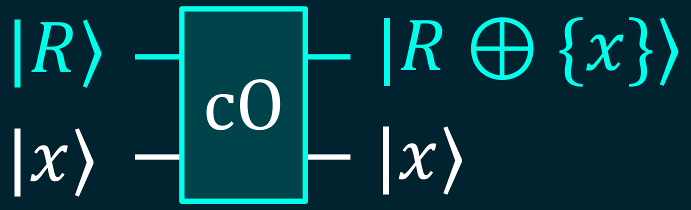

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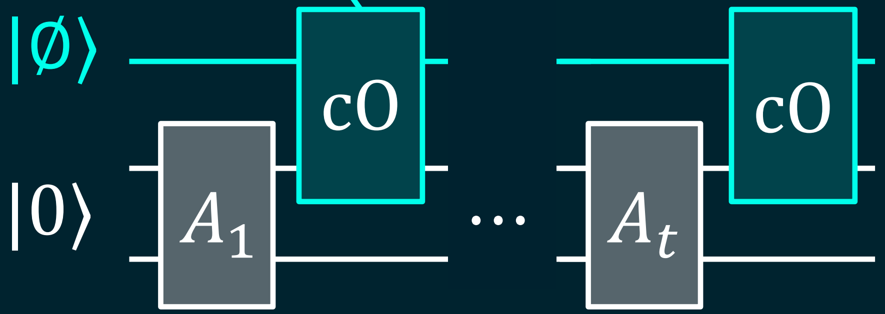

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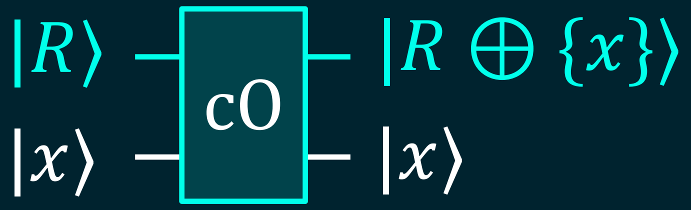

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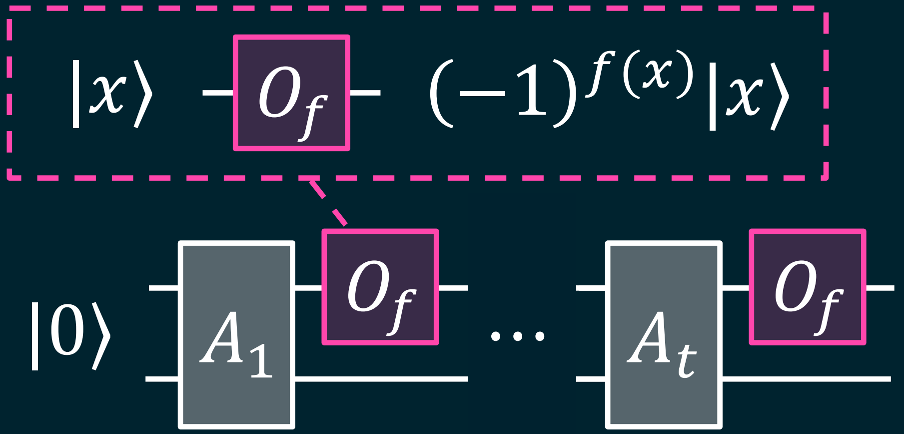

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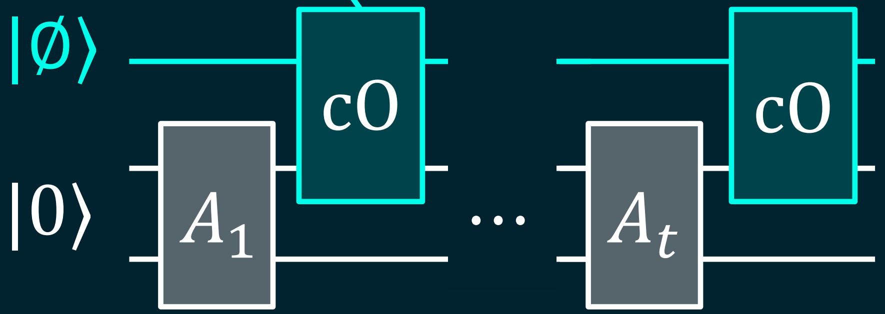

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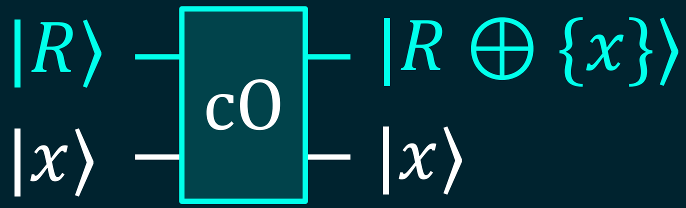

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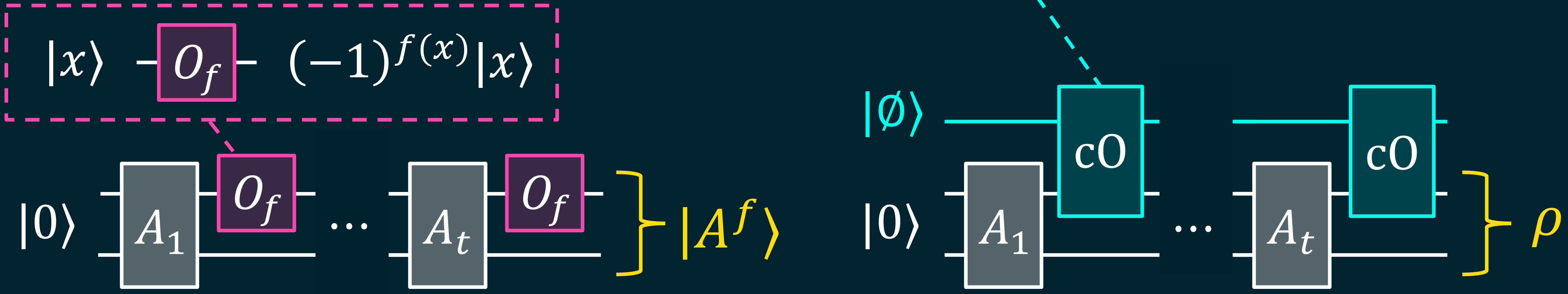

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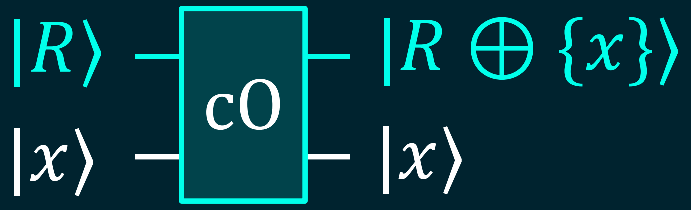

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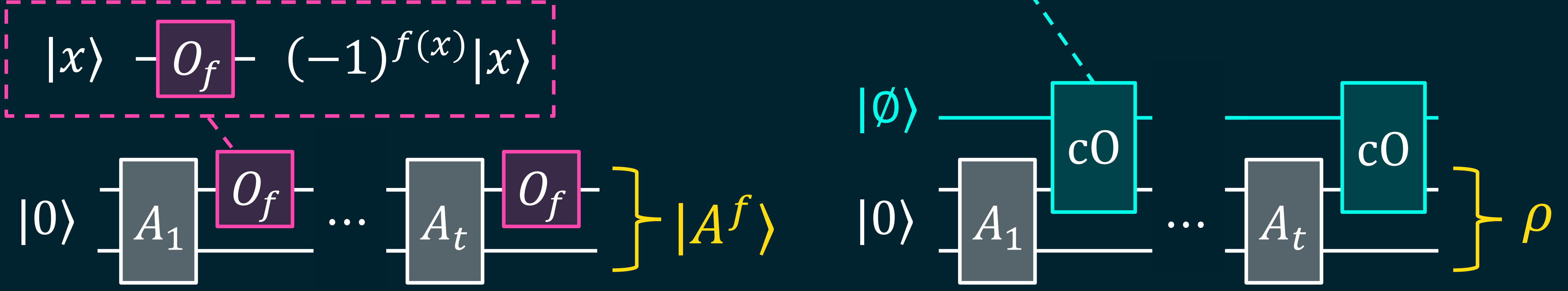

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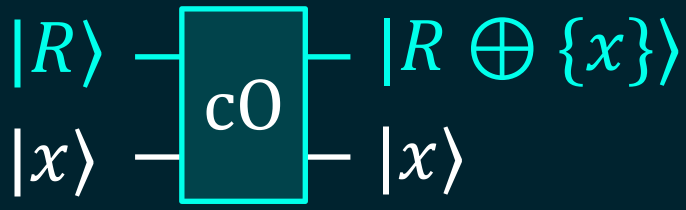

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and view $|f\rangle$ in the Fourier basis.

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Up next: the path-recording oracle

(our simulator for Haar-random unitaries)

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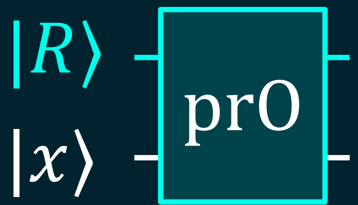

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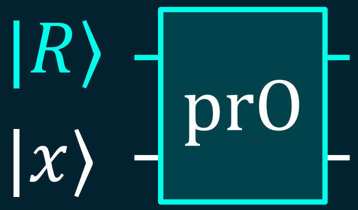

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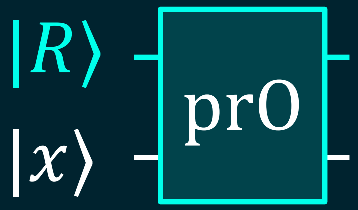

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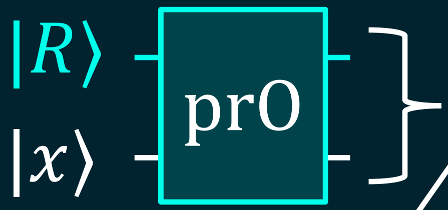

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Intuition: $|y\rangle |R \cup \{(x, y)\}\rangle$ uniquely determines $|x\rangle |R\rangle$.

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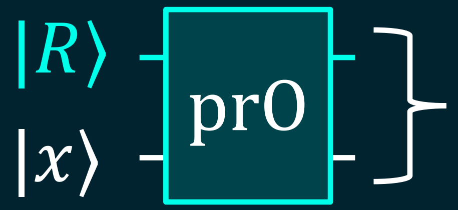

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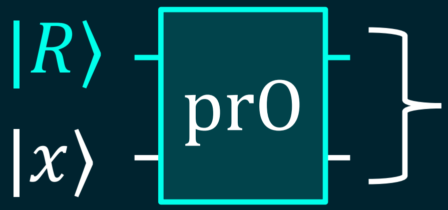

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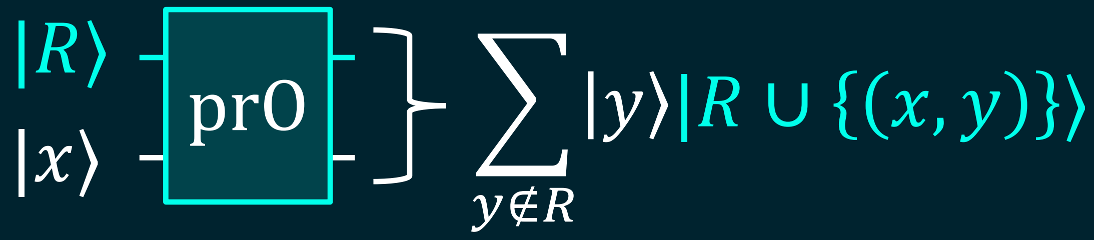

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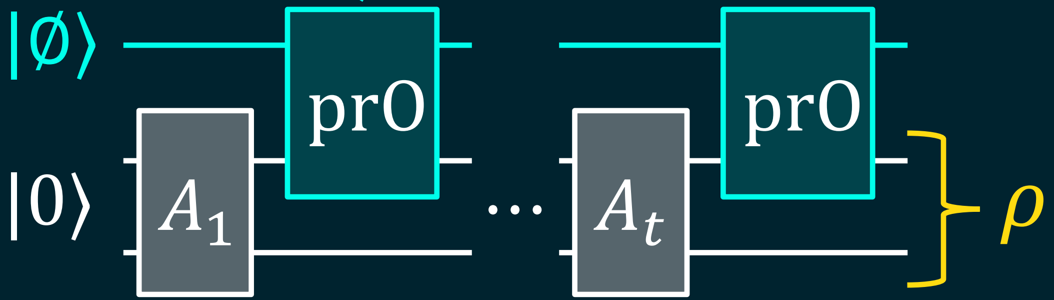

Efficient simulation

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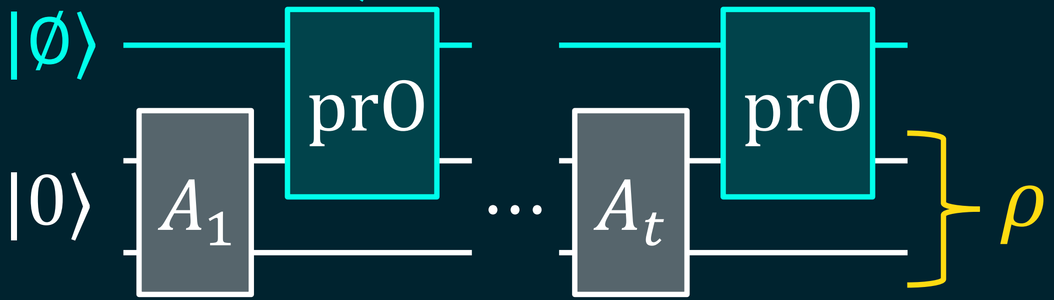

Efficient simulation

We show: $\mathbb{E}_{U \leftarrow \mathrm{Haar}} |A^U\rangle\langle A^U|$ and ρ have trace distance $\leq t^2/2^n$.

Up next: a few examples

Example 1: one query on $|0\rangle$

$$|0\rangle \;-\!\boxed{U}\!-\; U|0\rangle$$

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Average over $U \leftarrow \mathrm{Haar}$: $U|0\rangle$ becomes the *maximally mixed state.*

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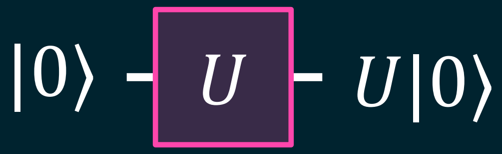

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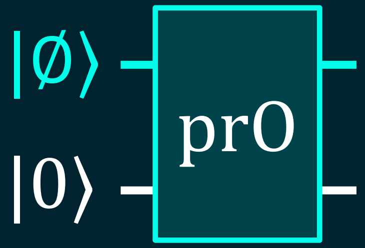

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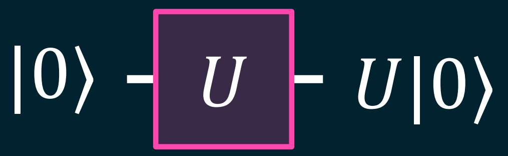

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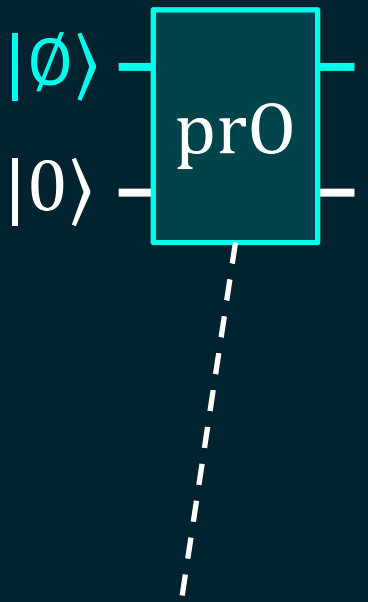

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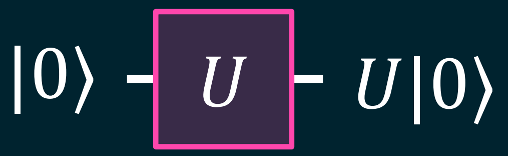

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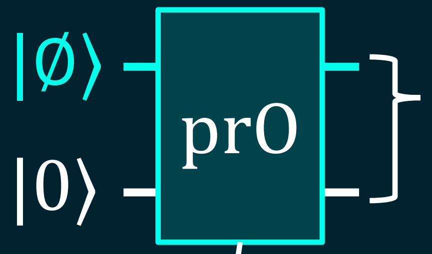

$\sum_y |y\rangle \otimes |\{(0, y)\}\rangle$

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$|0\rangle$ — $\boxed{U}$ — $U|0\rangle$

Average over $U \leftarrow \text{Haar}$: $U|0\rangle$ becomes the *maximally mixed state.*

$$\begin{array}{l} |\emptyset\rangle \\ |0\rangle \end{array} \boxed{\mathrm{prO}} \; \Sigma_y |y\rangle \otimes \underbrace{|\{(0,y)\}\rangle}_{\text{trace out/measure}}$$

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After tracing out: uniform mixture over $|y\rangle$, which is the *maximally mixed state.*

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Example 2: two queries on $|0\rangle$

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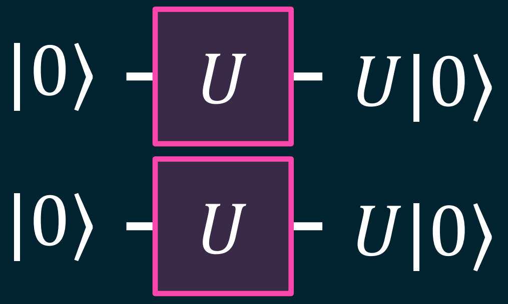

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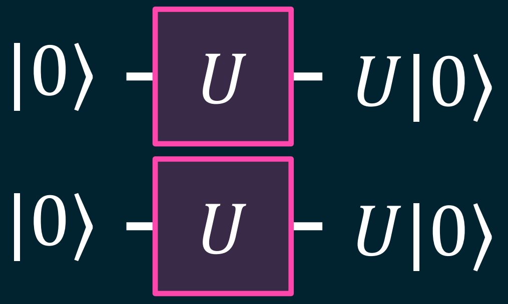

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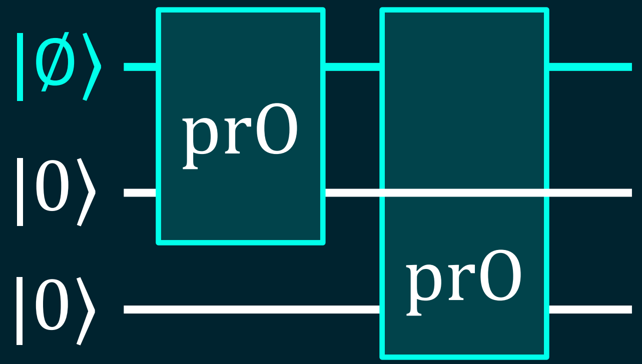

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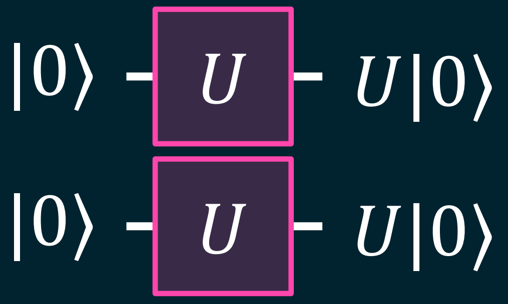

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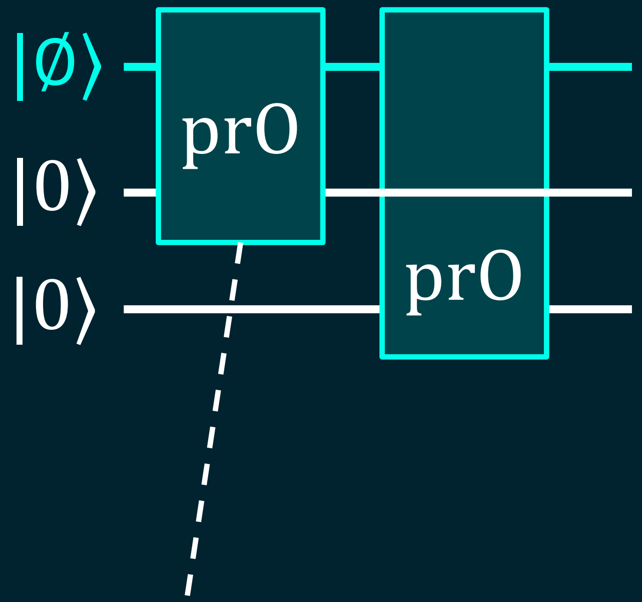

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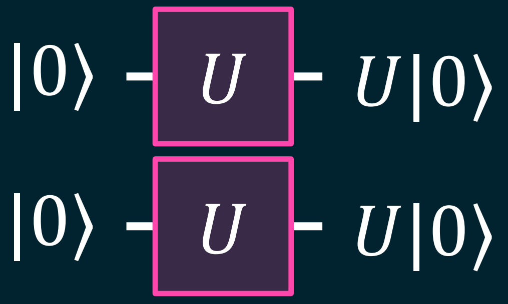

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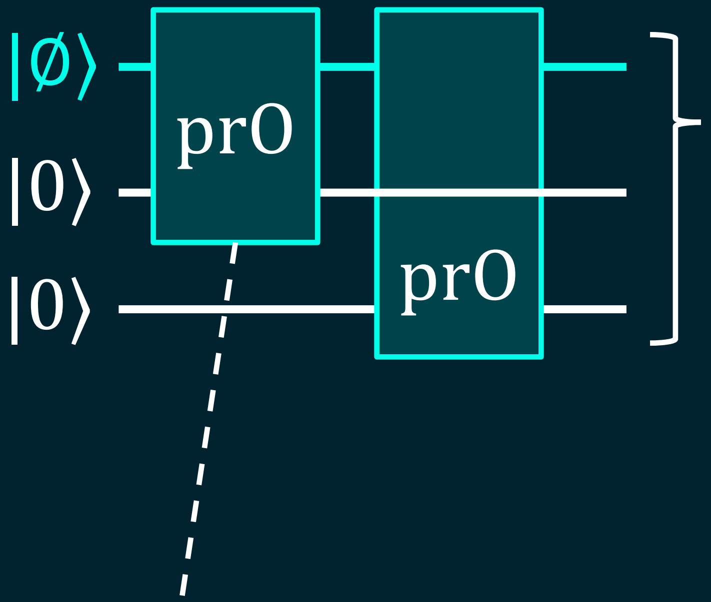

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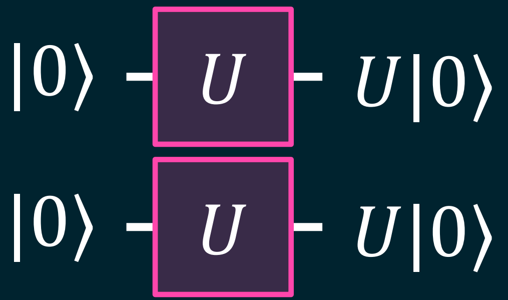

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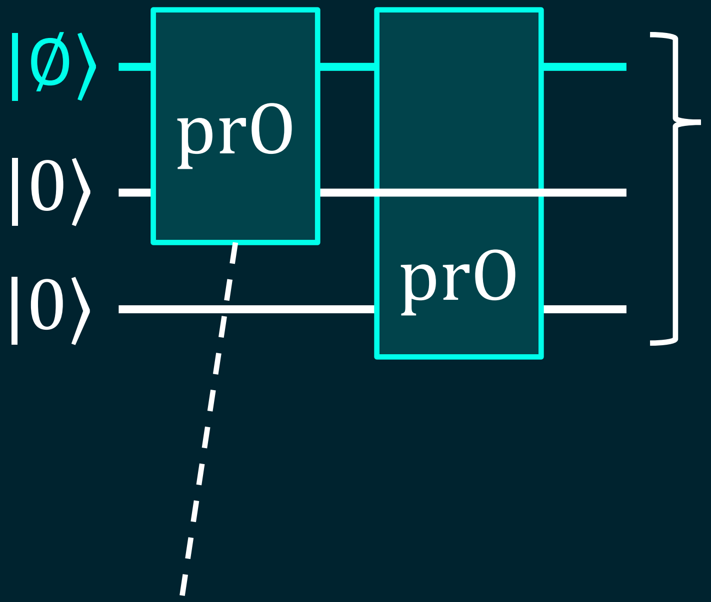

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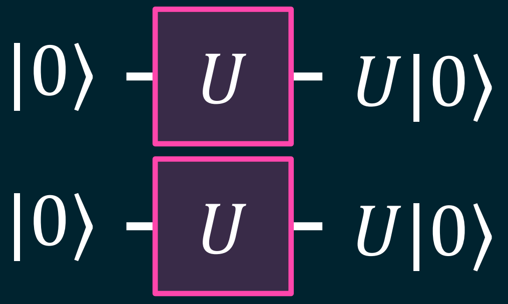

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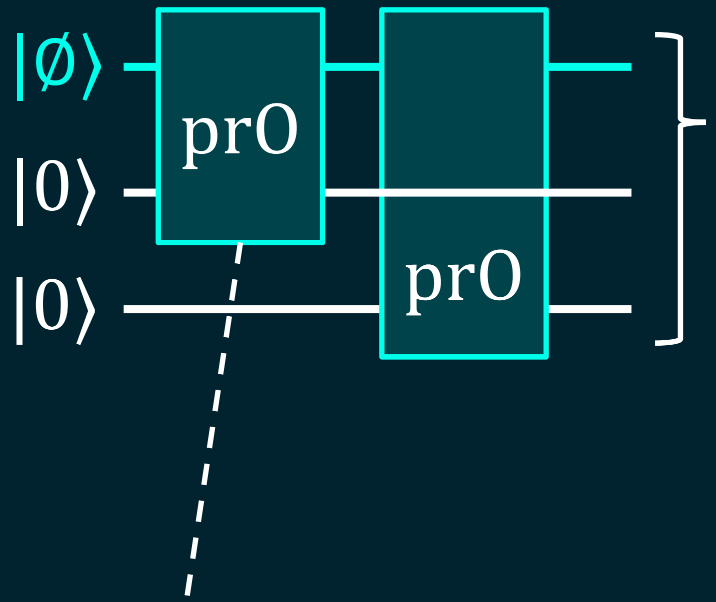

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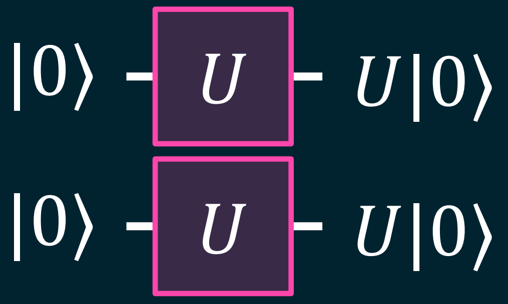

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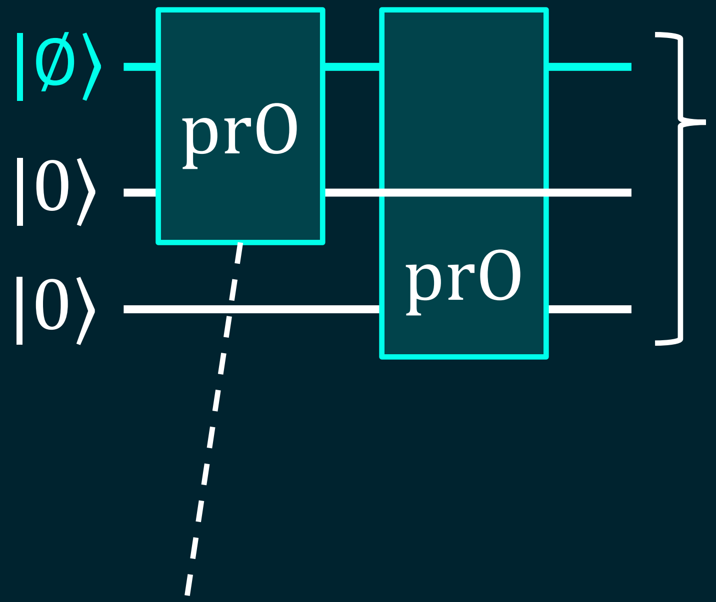

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After tracing out: mixture of $|y_1, y_2\rangle + |y_2, y_1\rangle$ for random distinct y_1, y_2. This is almost *maximally mixed on the symmetric subspace*.

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Up next:

Prove two claims simultaneously

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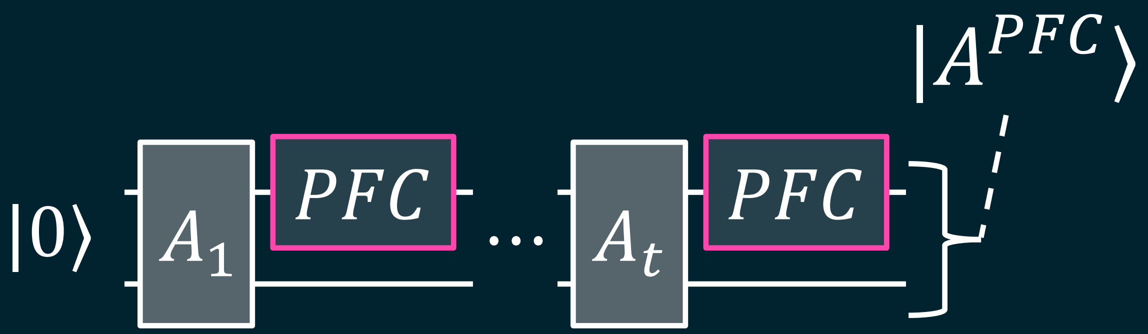

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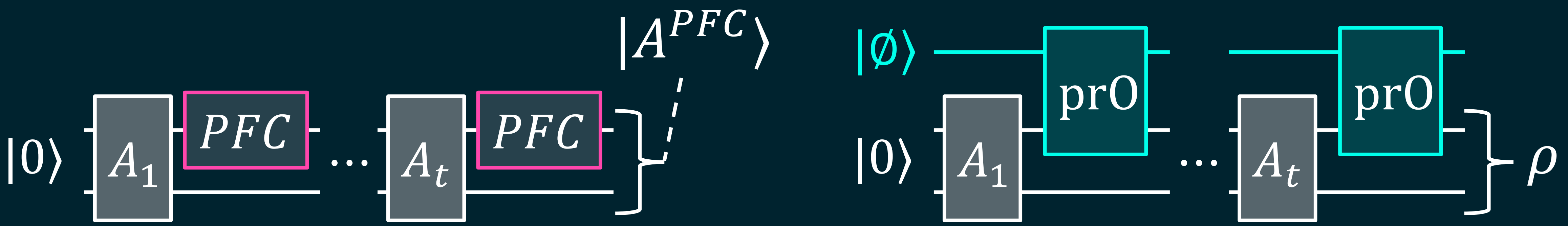

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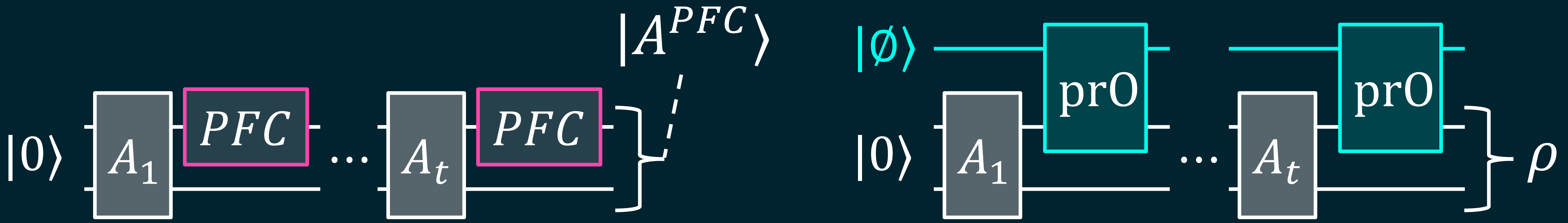

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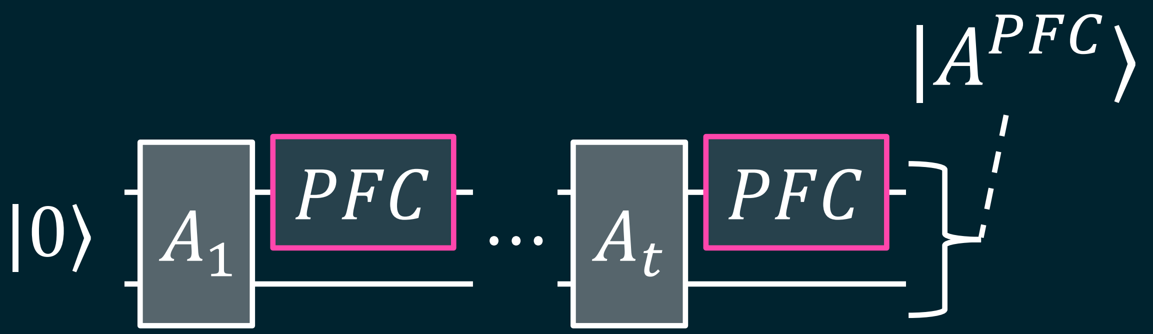

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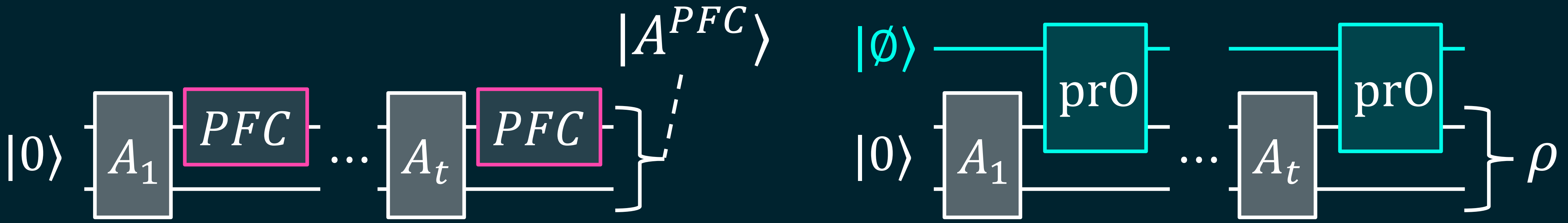

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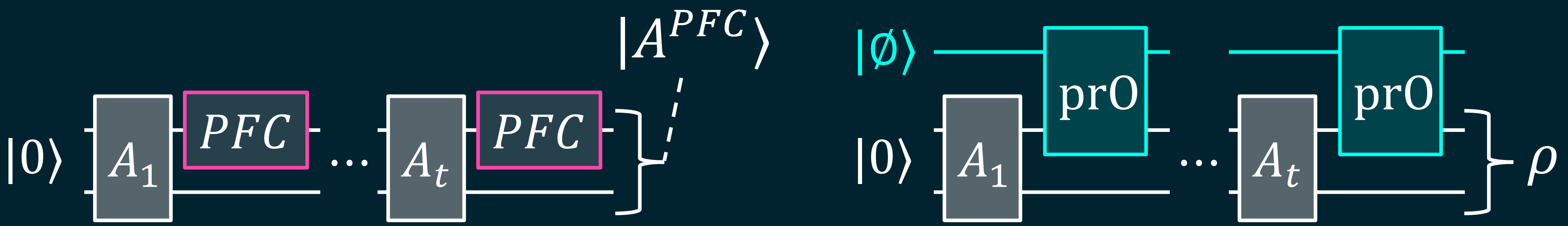

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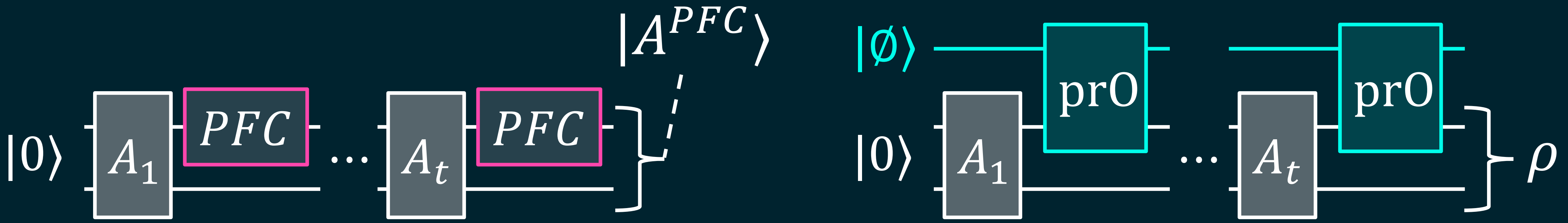

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Up next: proof of this claim

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Part 2: Insert a random C (sampled from any 2 design) to prevent A from querying on "bad" inputs.

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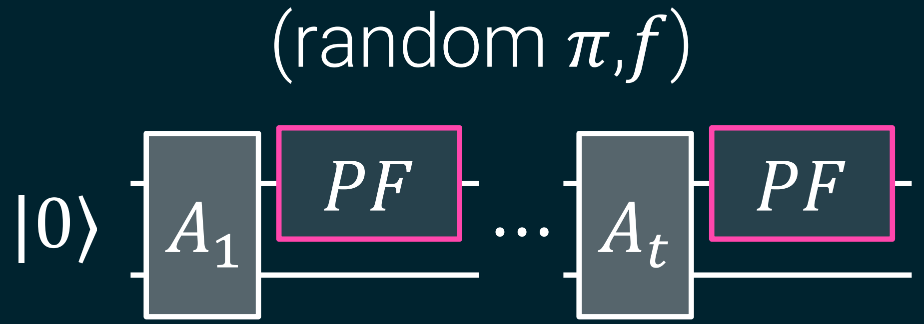

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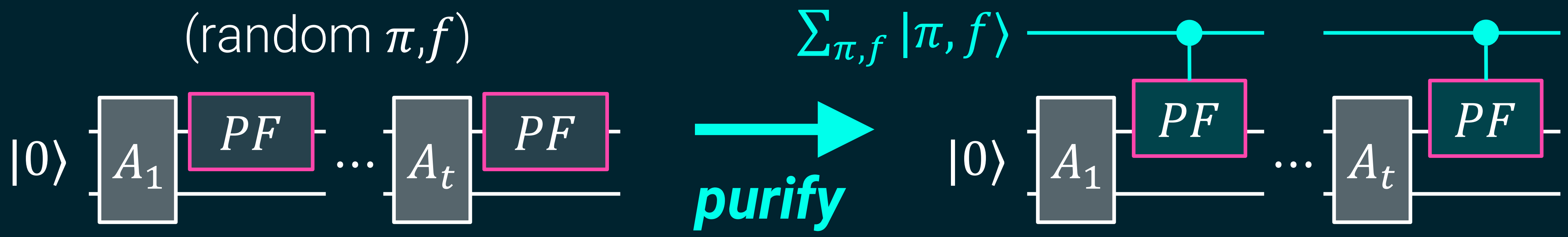

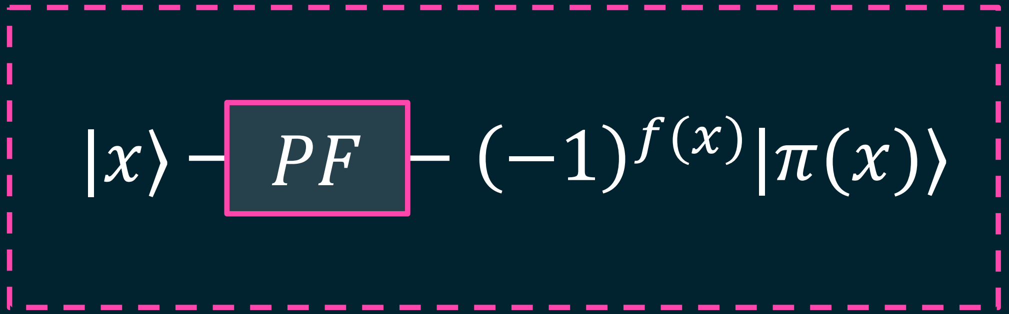

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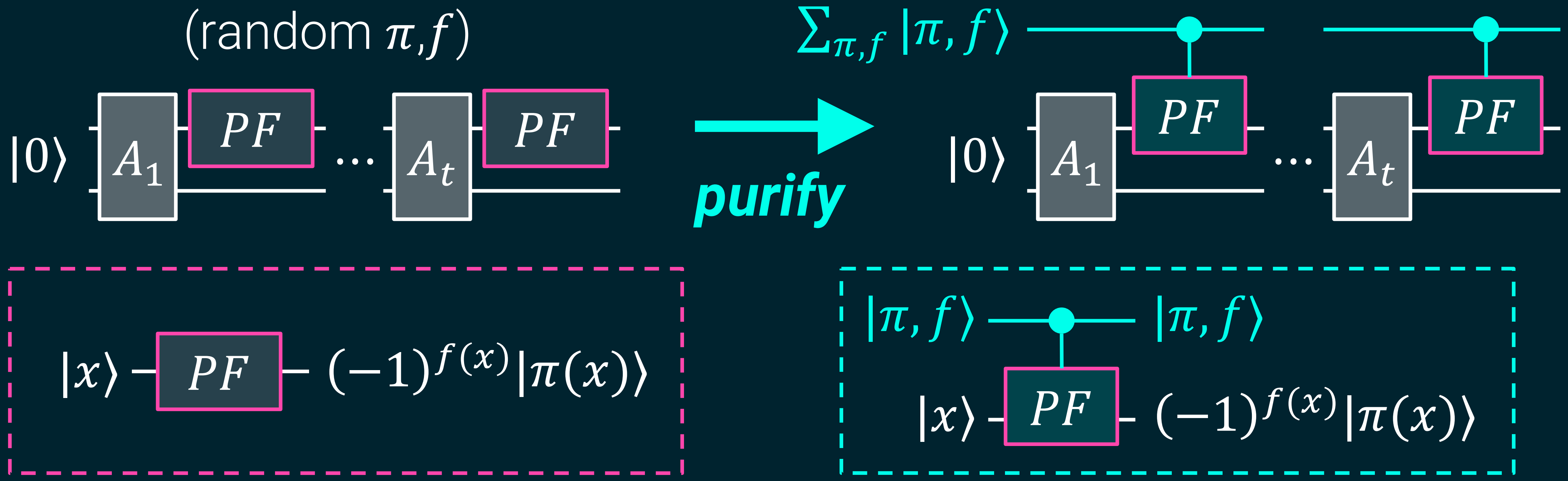
standard implementation | purification



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We'll show that controlled-PF does (almost) the same thing.

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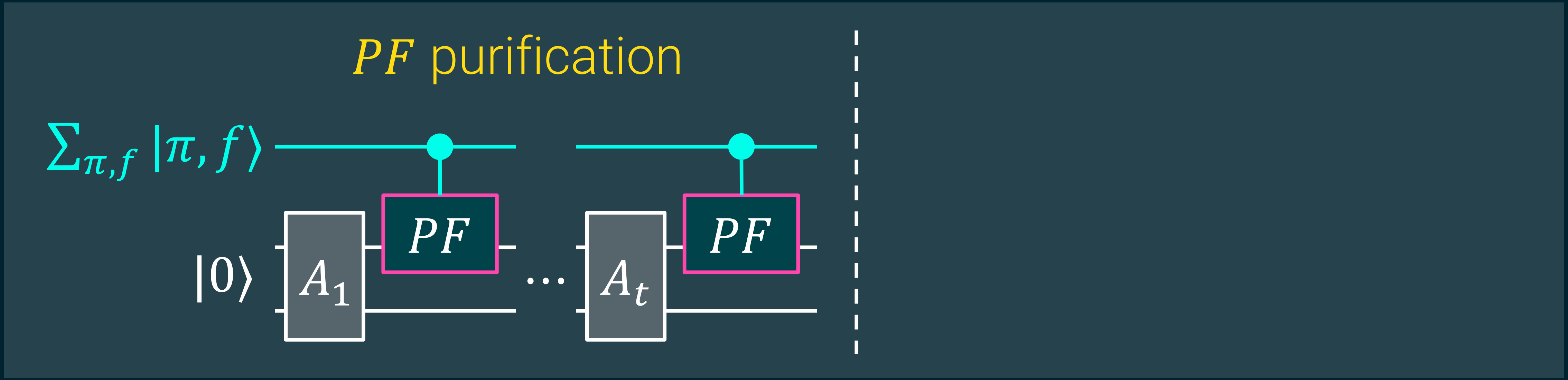

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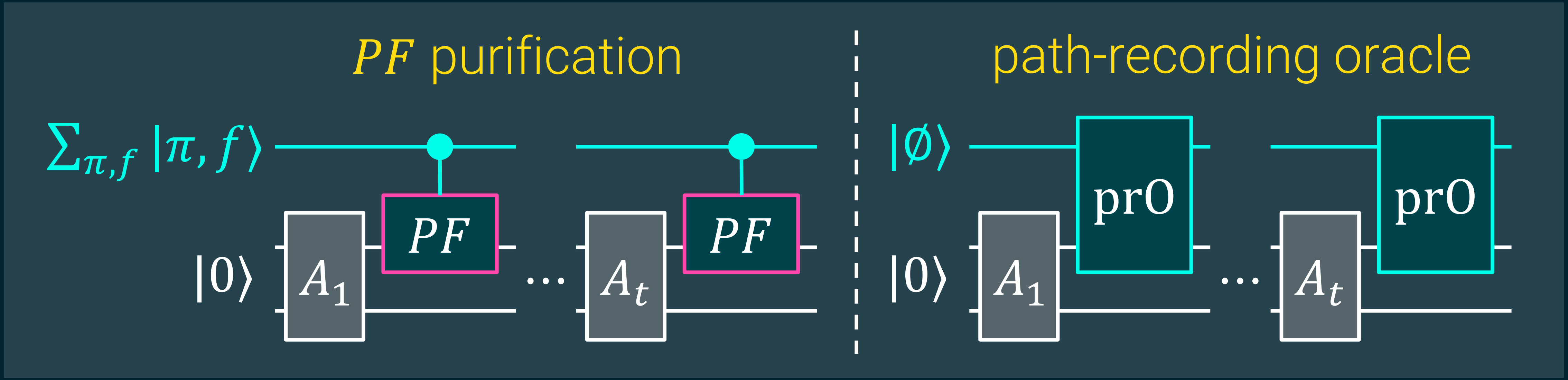

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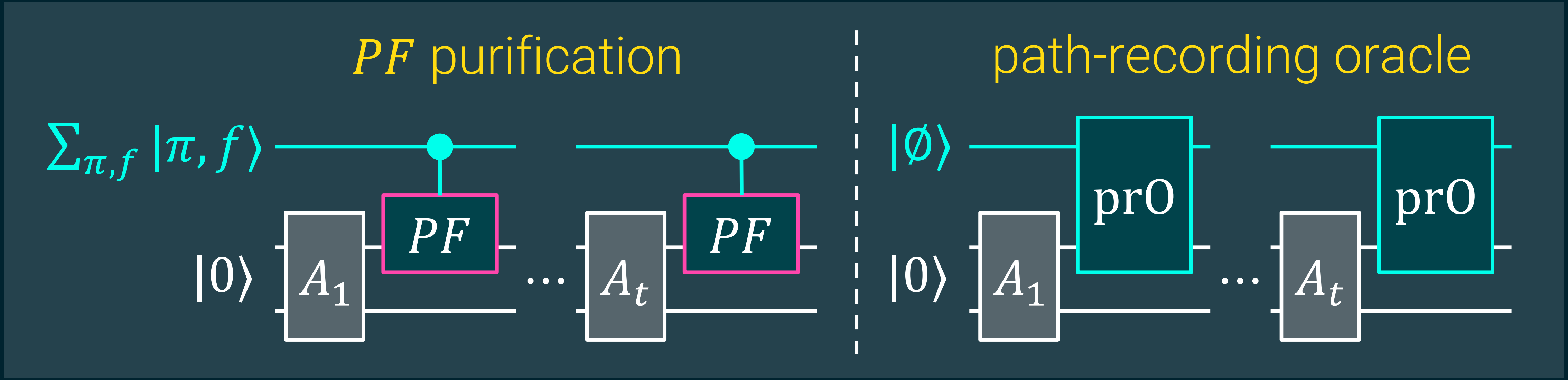

...unless $R = \{(x_1, y_1), \ldots, (x_t, y_t)\}$ has **colliding** x_i's.

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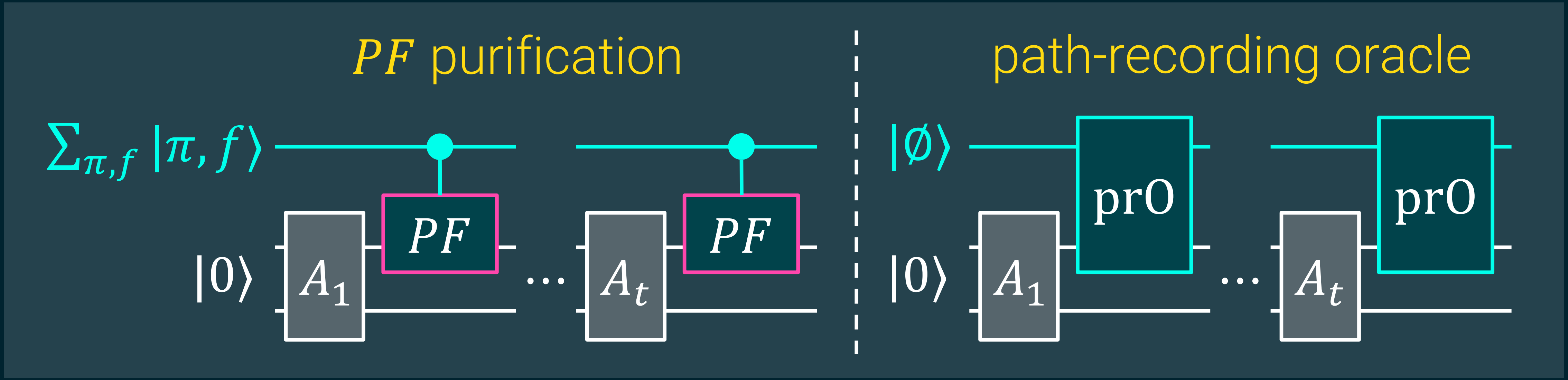
So these are equivalent from the algorithm's point of view:



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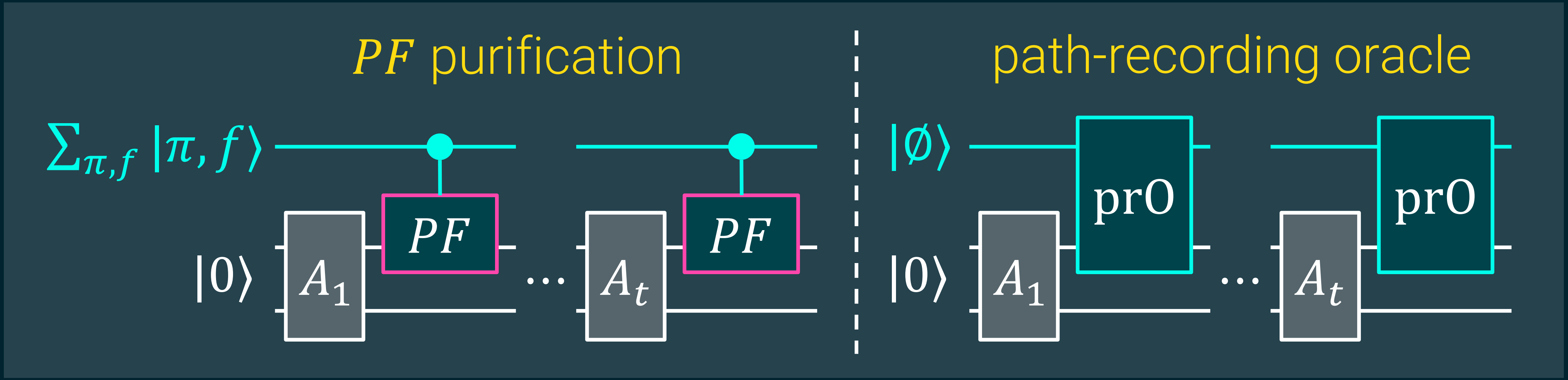

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Claim: inserting C **before** each query prevents collisions.

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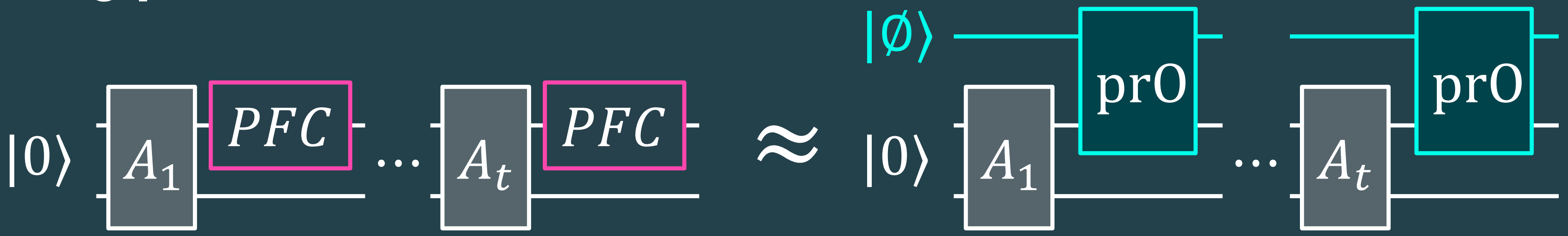

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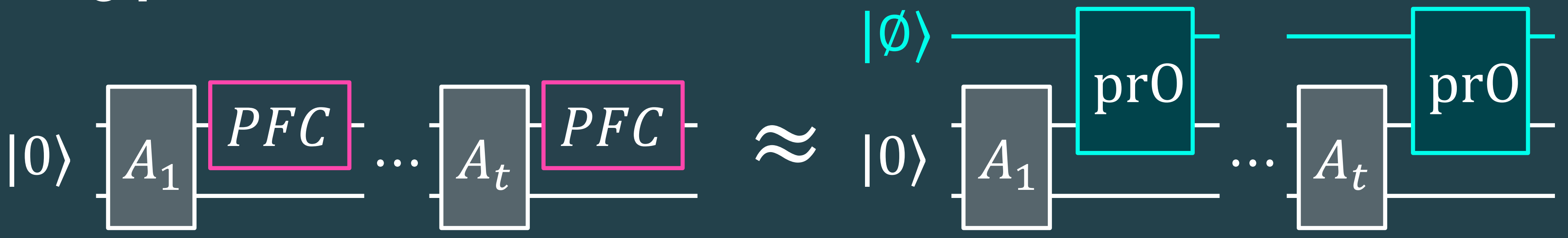

Consequence: prO simulates Haar-random unitaries + PRU exist

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Let’s see an example.

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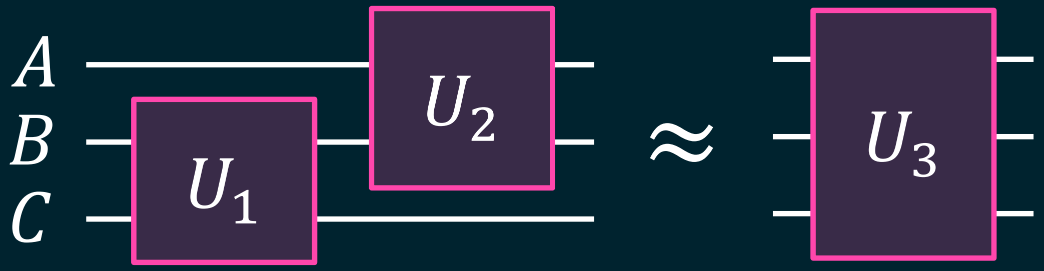

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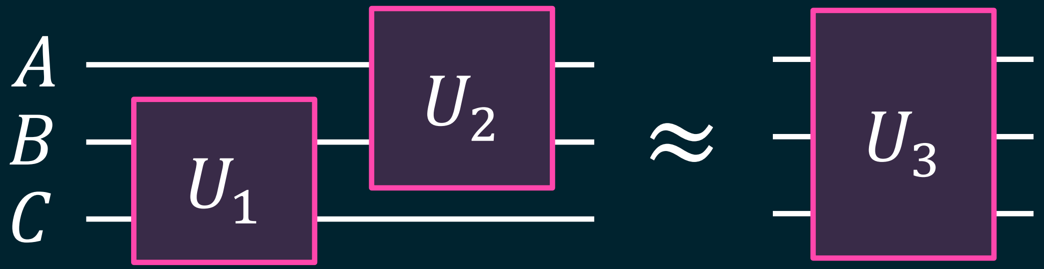

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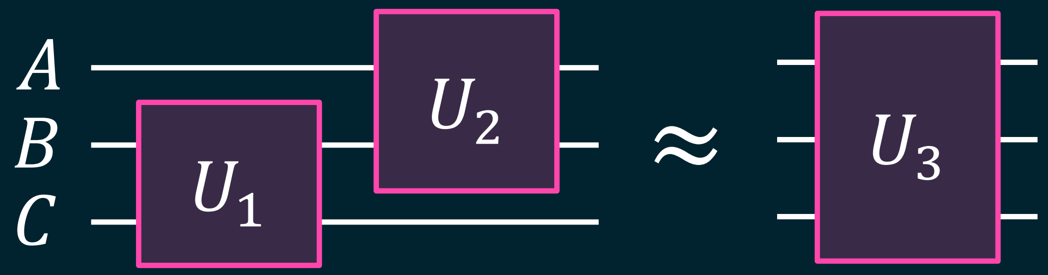

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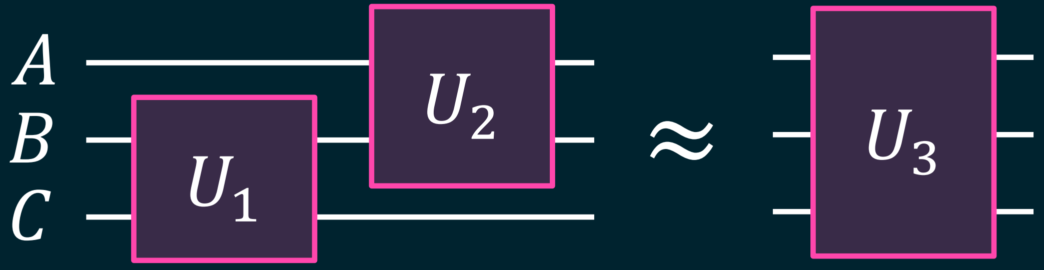

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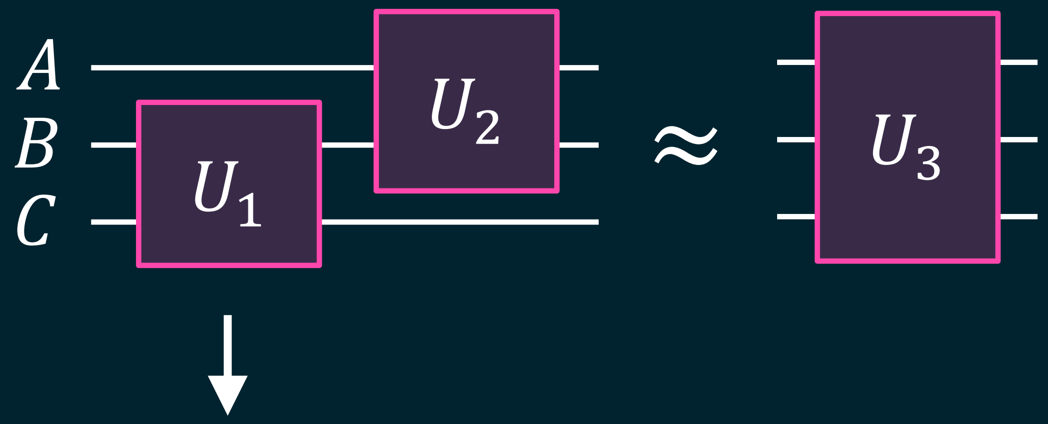

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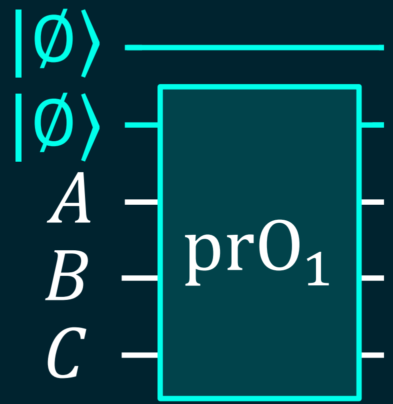

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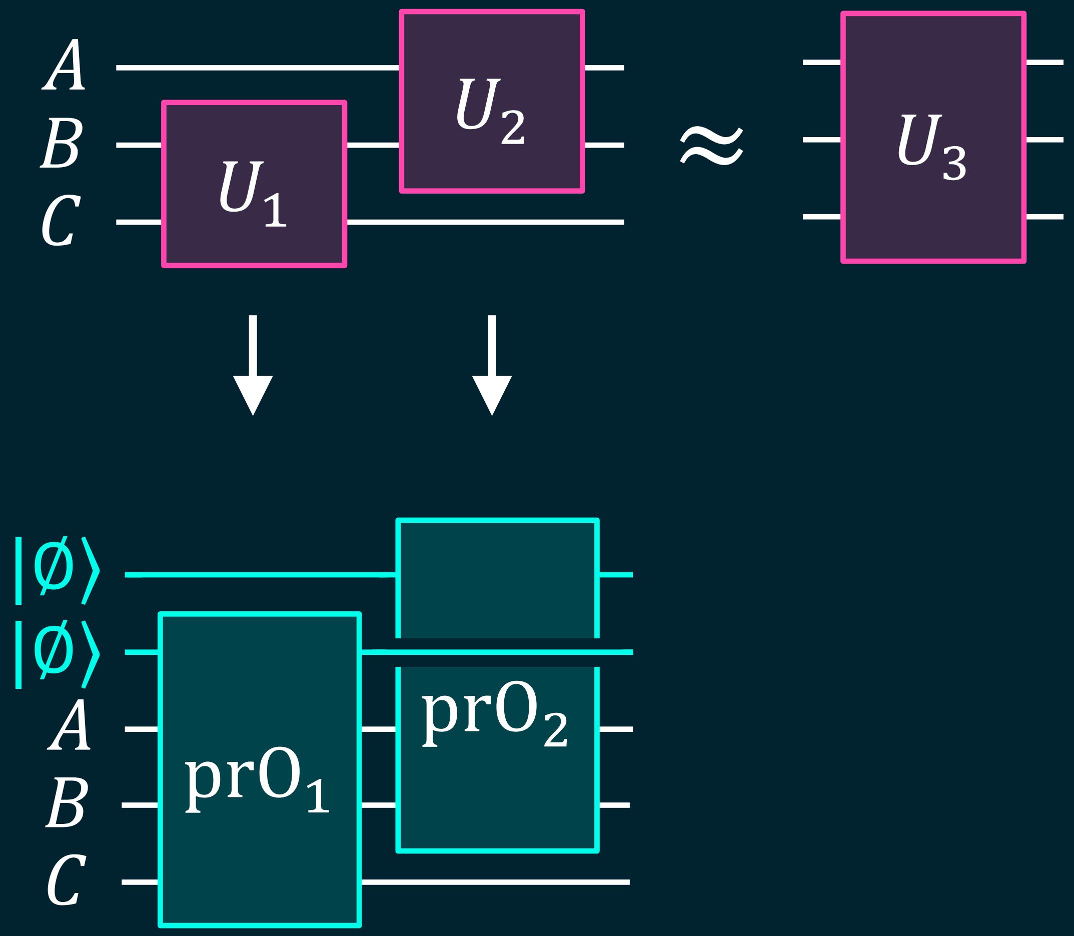

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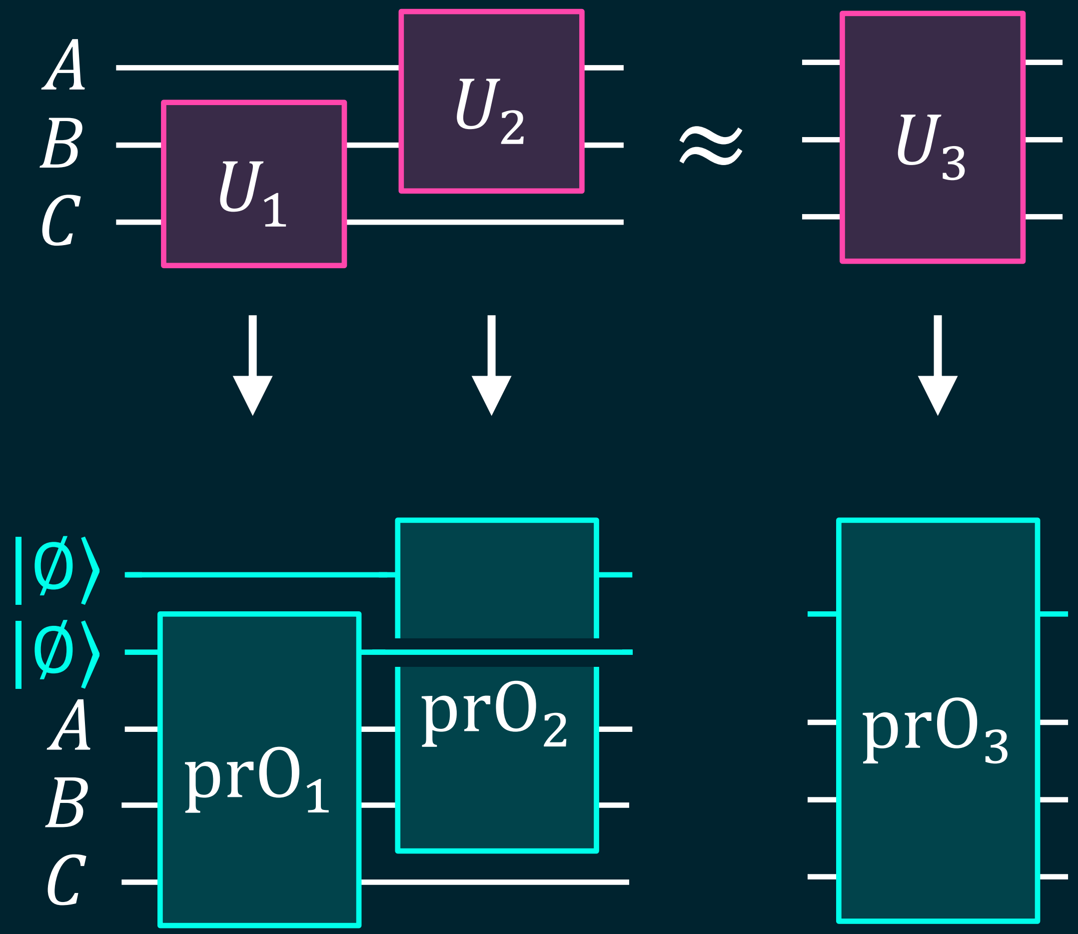

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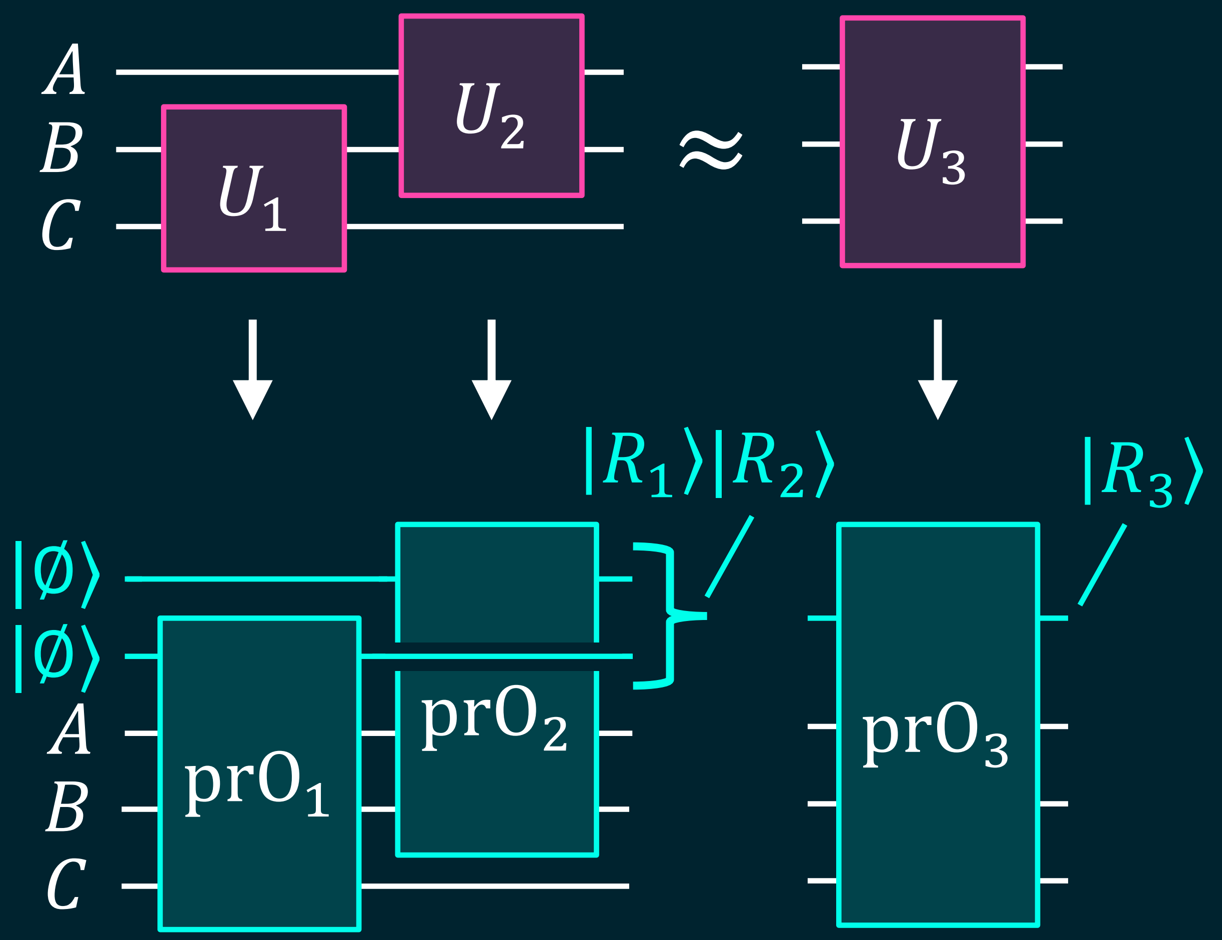

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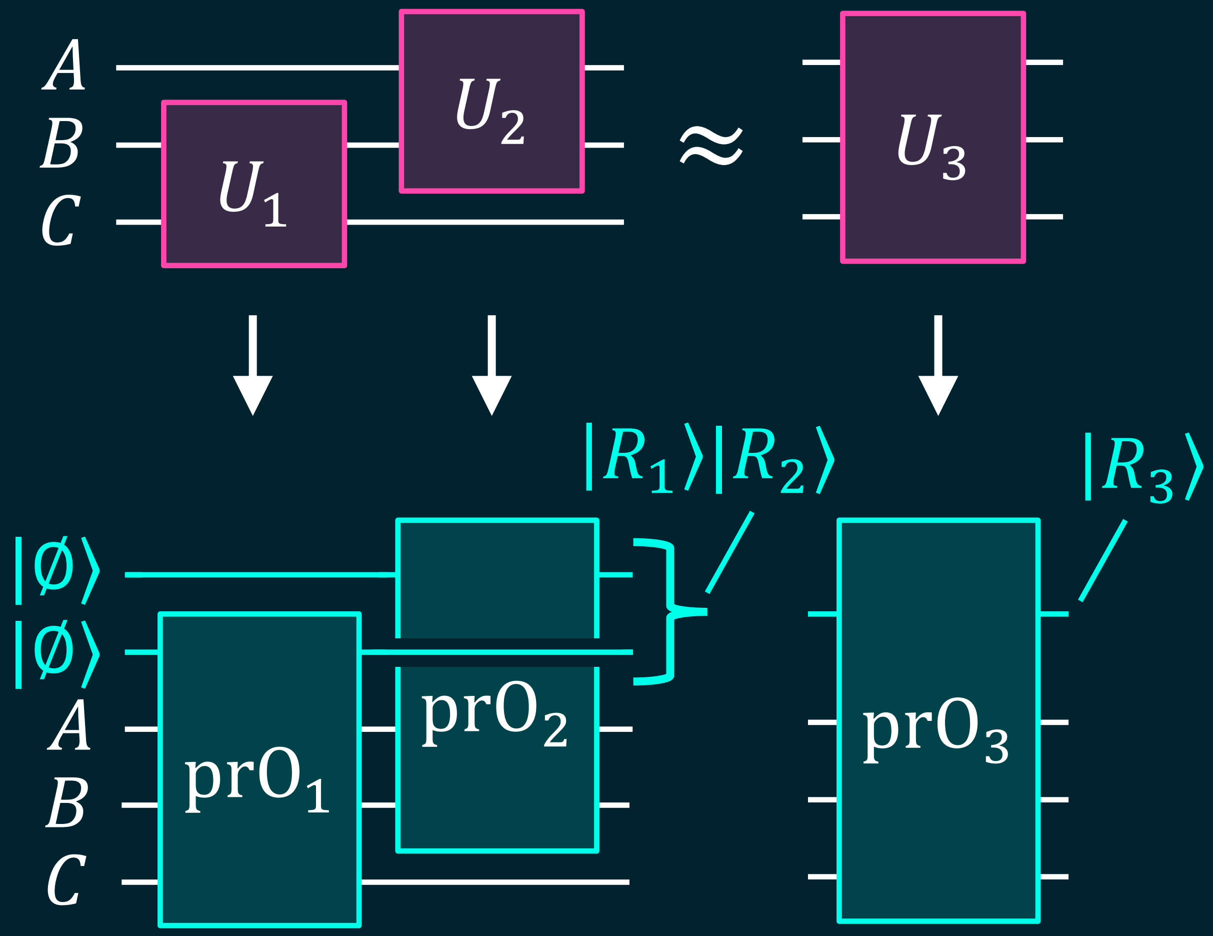

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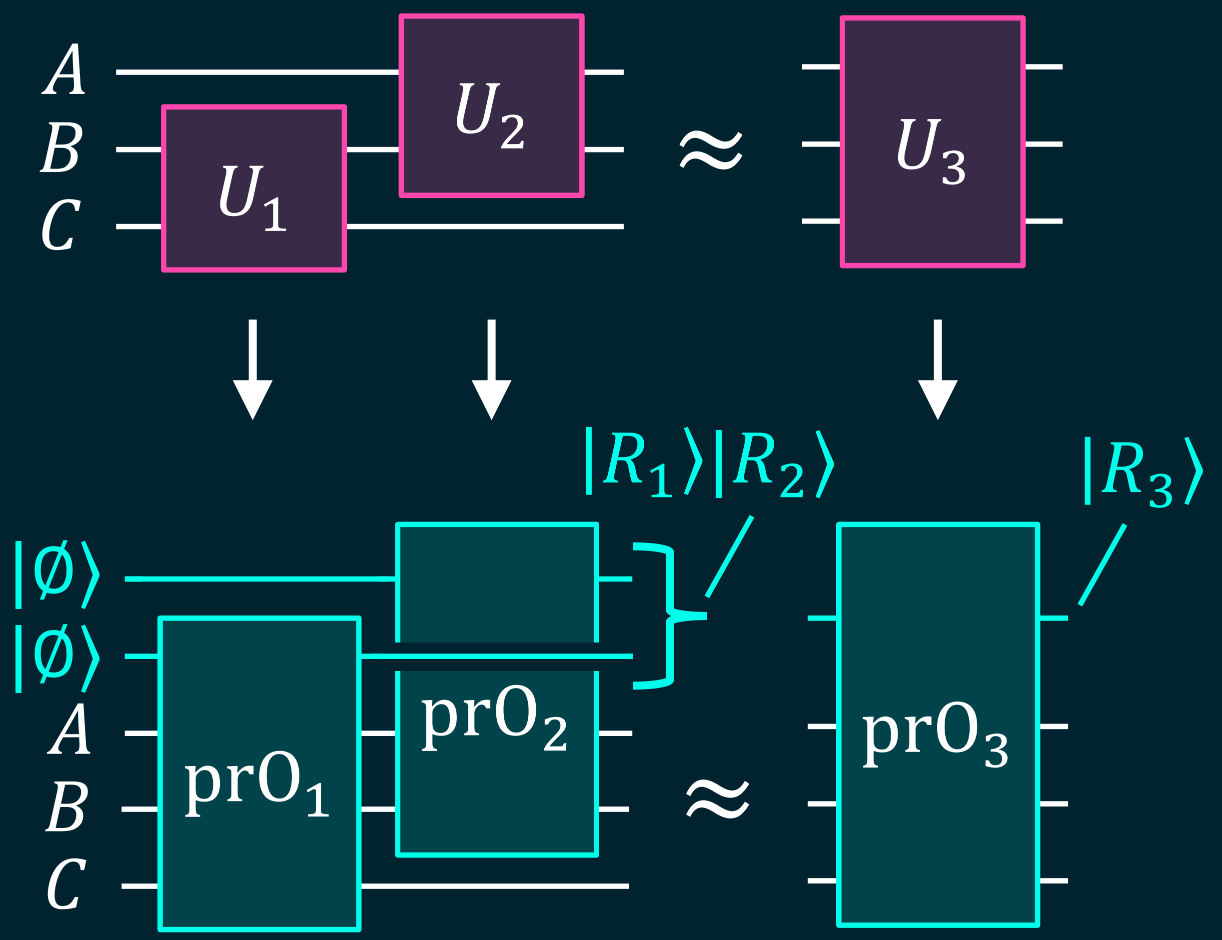

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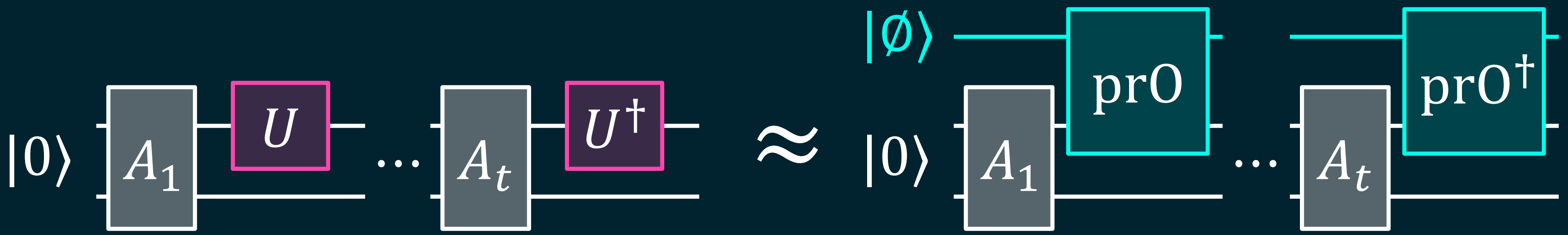

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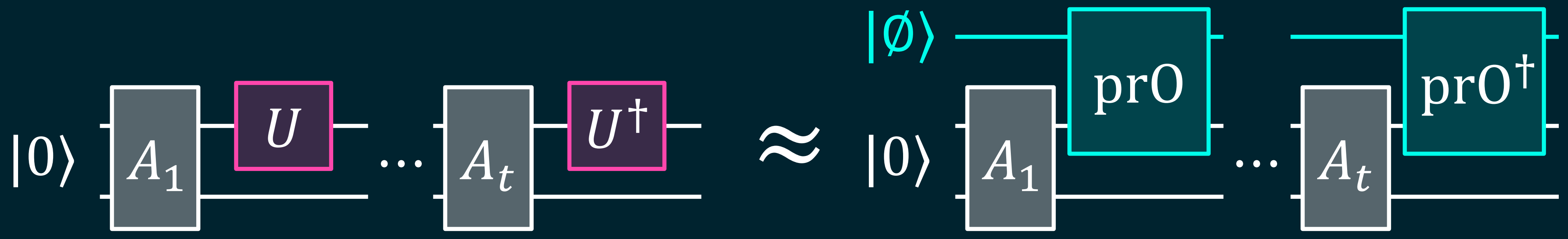

Challenge: recording the path isn't enough; also need to **erase**!

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Thanks for listening!

arXiv: 2410.10116